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Relativistic resonances, semigroup representation of Poincaré transformations, the exponential decay law and deviations thereof*

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Abstract

Starting from the phenomenological lineshape, relativistic Gamow vectors are defined. They span an irreducible representation ($[j, \mathsf{s}_R]$) of the causal Poincaré semigroup. Their transformation properties are presented, from which follow the exponential time evolution of the (relativistic) Gamow states (of spin j and mass $\mathsf{s}_R = (M - \mathrm{i}\Gamma/2)^2$). The preparation and analysis of decay data are complicated by the presence of a continuous background integral—always omitted in the Weisskopf–Wigner approximation—in the complex basis vector expansion for the in-state of a resonance scattering experiment. This background integral, which is related to the background term in the scattering amplitude, gives rise to deviations from the exponential time evolution. To what extent a prepared state decays exponentially depends on the experiment and not the resonance *per se*.

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1. Introduction

Resonances and decaying particles are characterized by two numbers, either by the resonance mass M_R (or resonance energy E_R) and the Breit–Wigner width Γ or by the resonance mass M and the decay rate (inverse of the lifetime $1/\tau$). In non-relativistic quantum mechanics, it is always assumed that $\Gamma = 1/\tau$, though this is only based on the Weisskopf–Wigner approximation. For relativistic resonances, opinions are more divided. Based on the perturbation theoretical definition of the self-energy of the propagator [1], resonances and decaying states are considered as complicated objects that cannot be described simply as an exponentially decaying state or as a state characterized by two numbers such as (M_R, Γ).

* To the memory of Lochlainn O'Raifeartaigh in gratitude and sorrow.

For the sake of definiteness, we want to consider a particular example for which the lineshape has been discussed in terms of the conventional field theoretical arguments. This is the Z-boson resonance in the process

$$e^- + e^+ \to Z \to f + \bar{f} \tag{1.1}$$

where $f\bar{f}$ is any of the lepton–antilepton or quark–antiquark pairs. This has been recently measured with very high precision [2–5]. The jth partial wave amplitude in this resonance formation process $a_j(\mathsf{s})$ is a function of the invariant mass square $\mathsf{s} = \left(p_1^\mu + p_2^\mu\right)^2 = \left(E_1^{cm} + E_2^{cm}\right)^2$, where p_1^μ, p_2^μ are the momenta of the two incoming (or outgoing) particles. If there is one resonance in the jth partial wave then one writes the amplitude of this resonance formation scattering process as

$$a_j(\mathsf{s}) = a_j^{\mathrm{res}}(\mathsf{s}) + B_j(\mathsf{s}). \tag{1.2}$$

Here $B_j(\mathsf{s})$ is a slowly varying background function and a_j^{res} is to describe the contribution of the resonance *per se*. In conventional quantum field theory, the amplitude contributing to the resonance $a_j^{\mathrm{res}}(\mathsf{s})$ is defined from the Z-boson propagator in the on-shell renormalization scheme and given by a Breit–Wigner with energy-dependent width[1]:

$$a_j^{\mathrm{res}}(\mathsf{s}) = a_j^{om}(\mathsf{s}) = \frac{-\sqrt{\mathsf{s}}\sqrt{\Gamma_e(\mathsf{s})\Gamma_f(\mathsf{s})}}{\mathsf{s} - M_Z^2 + \mathrm{i}\sqrt{\mathsf{s}}\Gamma_Z(\mathsf{s})} \approx \frac{-M_Z B_e B_f \Gamma_Z}{\mathsf{s} - M_Z^2 + \mathrm{i}\frac{\mathsf{s}}{M_Z}\Gamma_Z} = \frac{R_Z}{\mathsf{s} - M_Z^2 + \mathrm{i}\frac{\mathsf{s}}{M_Z}\Gamma_Z}. \tag{1.3}$$

$\Gamma_Z(\mathsf{s})$ is called the energy-dependent width[2] and as width one quotes in [2] $\Gamma_Z\left(\mathsf{s} = M_Z^2\right)$. When it was noticed that the on-shell mass and width definitions were gauge dependent [6–8] and arbitrary [9], a definition of mass and width by the complex pole of the propagator was proposed. It is not possible to fix the functions $a_j^{\mathrm{res}}(\mathsf{s})$ and $B_j(\mathsf{s})$ separately from the empirical data for $|a_j(\mathsf{s})|^2$ (besides the practical problems of determining $a_j(\mathsf{s})$ from the experimental data), unless one has some theoretical arguments in favour of a particular functional form of $a_j^{\mathrm{res}}(\mathsf{s})$ (or of $B_j(\mathsf{s})$).

The definition of the resonance by a pole of the propagator[3] at $s = \mathsf{s}_R$ leads to a unique function of s, which we call the relativistic Breit–Wigner,

$$a_j^{\mathrm{BW}}(\mathsf{s}) = \frac{r_Z}{\mathsf{s} - \mathsf{s}_R} = \frac{r_Z}{\mathsf{s} - \bar{M}_Z^2 + \mathrm{i}\bar{M}_Z\bar{\Gamma}_Z} = \frac{r_Z}{\mathsf{s} - \left(M_R - \mathrm{i}\frac{\Gamma_R}{2}\right)^2}. \tag{1.4}$$

This Breit–Wigner still allows for many different definitions of the real parameters, mass and width, of which $(\bar{M}_Z, \bar{\Gamma}_Z)$, (M_R, Γ_R), (m_1, Γ_1) are the best known parametrizations of s_R. They are

1. $(\bar{M}_Z, \bar{\Gamma}_Z)$ (also called (m_2, Γ_2)) [8] is defined as

$$\mathsf{s}_R = \bar{M}_Z^2 - \mathrm{i}\bar{M}_Z\bar{\Gamma}_Z. \tag{1.5}$$

It is the peak position of the relativistic Breit–Wigner $\left|a_j^{\mathrm{BW}}(\mathsf{s})\right|^2$.

2. (M_R, Γ_R) [6], which is suggested (but not dictated) by the analyticity of the S-matrix as a function of the Mandelstam variable s, is defined as

$$\mathsf{s}_R = \left(M_R - \mathrm{i}\frac{\Gamma_R}{2}\right)^2. \tag{1.6}$$

[1] See, e.g., section 35 of [2].

[2] The energy-dependent width is given in terms of the self-energy $\Pi(\mathsf{s})$ by $\sqrt{\mathsf{s}}\Gamma_Z(\mathsf{s}) = -\frac{\mathrm{Im}\Pi(\mathsf{s})}{1-\mathrm{Re}\Pi'(M_Z^2)} \approx \frac{\mathsf{s}}{M_Z}\Gamma_Z$ and the partial width is $\Gamma_f(\mathsf{s}) = B_f\Gamma(\mathsf{s})$ where B_f is the constant branching fraction into $f\bar{f}$.

[3] Or equivalently by a pole of the S-matrix. See section 3 below.

It is related to $(\bar{M}_Z, \bar{\Gamma}_Z)$ by

$$\bar{M}_Z = M_R \left(1 - \frac{1}{4}\left(\frac{\Gamma_R}{M_R}\right)^2\right)^{\frac{1}{2}} \tag{1.7}$$

$$\bar{\Gamma}_Z = \Gamma_R \left(1 - \frac{1}{4}\left(\frac{\Gamma_R}{M_R}\right)^2\right)^{-\frac{1}{2}}. \tag{1.8}$$

3. (m_1, Γ_1) [8] which can be defined in terms of $(\bar{M}_Z, \bar{\Gamma}_Z)$ by

$$m_1 = \sqrt{\bar{M}_Z^2 + \bar{\Gamma}_Z^2} \tag{1.9}$$

$$\Gamma_1 = \frac{m_1}{\bar{M}_Z}\bar{\Gamma}_Z \tag{1.10}$$

and which was suggested because (m_1, Γ_1) came closest in value to the definition (M_Z, Γ_Z) by (1.3). The association of the quasistable particle with the complex pole of the S-matrix (1.4) does not specify any particular separation of the complex pole position s_R into mass and width.

From the lineshape data $|a_j(\mathsf{s})|^2$ for the Z-boson, one cannot distinguish between $a_j^{\mathrm{BW}}(\mathsf{s})$ and $a_j^{om}(\mathsf{s})$ because one can shift the difference between them into the slowly varying function $B_j(\mathsf{s})$ of (1.2). Both (1.3) and (1.4) give equally good fits to the experimental lineshape data. However, they produce different values for the masses. From the fit of $a_j^{\mathrm{BW}}(\mathsf{s})$ one obtains [3–5],

$$M_R = 91.1626 \pm 0.0031\,\mathrm{GeV} \tag{1.11}$$

$$\Gamma_R = 2.4934 \pm 0.0024\,\mathrm{GeV} \tag{1.12}$$

and from $a_j^{om}(\mathsf{s})$

$$M_Z = 91.1871 \pm 0.0021\,\mathrm{GeV} \tag{1.13}$$

$$\Gamma_Z = 2.4945 \pm 0.0024\,\mathrm{GeV}. \tag{1.14}$$

For mass values quoted with this accuracy, the difference between the on-shell mass M_Z and the pole masses M_R and also $\bar{M}_Z$ is significant: $M_Z - M_R = 0.025$ MeV $= 10 \times$ experimental error, $\Gamma_Z - \Gamma_R \approx 1.1$ MeV. The values of (M_Z, Γ_Z) and $(\bar{M}_Z, \bar{\Gamma}_Z)$, $\bar{M}_Z = M_Z - 34.1$ MeV, are quoted in the PDT [2], the values of (M_R, Γ_R) are not.

To summarize, there is no answer to the question, what is the right definition of the mass and width of the Z-boson if one considers the lineshape only. Moreover, there is not even a phenomenological means to discriminate between the lineshape formulae of (1.3) and (1.4). The difference between them can be shifted to a non-resonant background term $B(\mathsf{s})$ of the scattering amplitude (1.2). For the ρ-meson resonance the experimental data give a slight preference to the S-matrix pole definition (1.4) if one makes the additional assumption that one and the same background function $B(\mathsf{s})$ should work for all channels [10].

Even after one has decided in favour of the relativistic Breit–Wigner (1.4) for the lineshape of the resonance, the question of the definition of resonance mass and width of a relativistic resonance is still completely open because one can parametrize the complex pole position s_R in many ways, of which (1.5), (1.6), (1.9) and (1.10) are the most popular. This question cannot be decided on the basis of the lineshape of a relativistic resonance alone. It is only possible within the framework of a broader theory that includes the time evolution of a resonance considered as a quasistationary state.

This will be the subject of the following section, where relativistic quasistationary states will be introduced in analogy with Wigner's theory for stable relativistic particles. To define this state we shall use the empirical form for the resonance part of the jth partial wave amplitude $a_j^{\text{res}}(\mathsf{s})$ given by the relativistic Breit–Wigner (1.4).

2. Semigroup representations of the Poincaré group for quasistable relativistic particles

Wigner's unitary irreducible representations (UIR) of the Poincaré group $\mathcal{P}$ provide the definition for relativistic stable particles [11]. They describe not only the free asymptotic states but also the interacting states [12]. These UIR are characterized by two quantum numbers: m^2 and j which are interpreted as the quantum number spin (j) and mass squared (m^2) of the relativistic stable particle. For the representation spaces $[j, m^2]$ one uses the basis vectors $|[j, m^2], b\rangle$ where b denotes the additional quantum numbers, for which one has various choices depending upon the complete set of commuting observables that one takes. For Wigner's canonical basis system the choice for b is the momentum and a component of the spin $\{\mathbf{p}, j_3\}$. One could have as well chosen $\{\hat{\mathbf{p}}, j_3\}$ with $\hat{\mathbf{p}} = \mathbf{p}/m$ being the spatial components of the four-velocity $\hat{p}$ [13][4]. It does not make any difference for the UIR of $\mathcal{P}$ whether one chooses for the degeneracy labels b the momentum $\mathbf{p}$ or the spatial component $\hat{\mathbf{p}}$ of the four-velocity. However, $\hat{\mathbf{p}}$ is the preferred choice for us here since we need to continue the invariant mass squared $\mathsf{s} = p_\mu p^\mu$ to complex values, when we analytically continue the S-matrix to the resonance pole position s_R. Complex mass will automatically lead to complex momenta, but it will not lead to complex four-velocity $\hat{p}$, since the Lorentz boost is a function of $\hat{p}_\mu$ and not of momentum p_μ.

A relativistic stable particle state characterized by m^2 and j, $f_{[j,m^2]}$, is the continuous superposition of the basis vectors $|[j, m^2], b\rangle$ with some measure μ:

$$f_{[j,m^2]} = \int \mathrm{d}\mu(b)|[j, m^2], b\rangle f(b). \tag{2.1}$$

In (2.1) the wavefunction of b, $f(b)$, is a well-behaved (Schwartz) function of b, $f(b) \in S(\mathbb{R}^3)$, and the measure is chosen to be Lorentz invariant, either $\mathrm{d}\mu(b) = \mathrm{d}^3\mathbf{p}/2\mathbf{p}^0$, or if one chooses $b = \hat{p}$, $\mathrm{d}\mu(b) = \mathrm{d}^3\hat{\mathbf{p}}/2\hat{\mathbf{p}}^0$. The integration in (2.1) is understood to comprise summation over all discrete degeneracy quantum numbers (e.g. j_3) encapsulated in b.

In a scattering process of two incoming particles, 1 and 2, and two outgoing particles, 3 and 4 as in (1.1), the system of out-observable vectors $\psi_3 \times \psi_4 \equiv \psi^-$, can be expressed in terms of out-state basis vectors, which span a two-particle irreducible representation of the Poincaré group. From the direct product basis of the two-particle space, one obtains new basis vectors using the Clebsch–Gordon coefficients of the Poincaré group[5]. These basis vectors are labelled by the total invariant mass square $s = (p_3 + p_4)^2$ and the total angular momentum j of the 3, 4 system and again by the degeneracy labels b and some other degeneracy labels such as total orbital angular momentum and total spin which we shall suppress here (they are irrelevant for the scattering of two spinless in- and out-particles). Thus the

[4] See also footnote 6 of this paper and references therein concerning the use of four-velocity kets.

[5] For the momentum basis vectors of the Poincaré group, these Clebsch–Gordon coefficients of the Poincaré group were discussed and calculated long ago [14]. They are needed for the relativistic partial wave analysis. For the velocity basis vectors, these Clebsch–Gordon coefficients have been derived in [13].

interaction-incorporating out-states ψ^- of a scattering experiment are continuous superpositions of the out-plane wave solutions of the Lippmann–Schwinger equation which we denote by $|[j, s], b^-\rangle$. Here $[j, \mathsf{s}]$ denotes the irreducible representation of the Poincaré transformations and for b we choose the four-velocity for the two-particle system, $\hat{\mathbf{p}} = (\mathbf{p}_3 + \mathbf{p}_4)/\sqrt{\mathsf{s}}$, the third component of the total angular momentum j_3 and possibly the discrete degeneracy quantum numbers or species labels n. An out-observable vector with a fixed value for s and j, $\psi^-_{[j,\mathsf{s}]}$, can thus be expanded in analogy with (2.1) in terms of $|[j, \mathsf{s}], \hat{\mathbf{p}}, j_3^-\rangle$ by the Dirac basis vector expansion,

$$\psi^-_{[j,\mathsf{s}]} = \sum_{j_3} \int \frac{\mathrm{d}^3\hat{\mathbf{p}}}{2\hat{\mathbf{p}}^0} |[j, \mathsf{s}], \hat{\mathbf{p}}, j_3^-\rangle \psi(\hat{\mathbf{p}}, j_3) \tag{2.2}$$

where $\psi(\hat{\mathbf{p}}, j_3)$ is a smooth function of $\hat{\mathbf{p}}$, $\psi(\hat{\mathbf{p}}) \in S(\mathbb{R}^3)$ as in (2.1). The general out-observable vector $\psi^- = \psi_3 \times \psi_4$ in the two-particle space is the continuous linear combination of the $\psi^-_{[j,\mathsf{s}]}$ over all j and s

$$\psi^- = \sum_j \int_{(m_3+m_4)^2}^{\infty} \mathrm{d}\mathsf{s}\, \psi^-_{[j,\mathsf{s}]} \psi_j(\mathsf{s}) \tag{2.3}$$

where $\psi_j(\mathsf{s})$ are wavefunctions of the total energy s. The expansion (2.3) expresses the reduction of the direct product of the representation $[j_3, m_3] \otimes [j_4, m_4]$ with respect to the irreducible representations $[j, \mathsf{s}]$ of the Poincaré group [13, 14]. Similar expansions to (2.2), (2.3) also hold for the in-state vector ϕ^+ of the scattering experiment in terms of the in-plane wave solutions of the Lippmann–Schwinger equation $|[j, \mathsf{s}], b^-\rangle$. The basis vectors $\left|[j, m^2], \hat{\mathbf{p}}, j_3^{\mp}\right\rangle$ are eigenvectors of the 'exact generators' of the Poincaré group. The exact generators are those which include an interaction such as $P_0 \equiv H = H_0 + V$ [12]. Of these eigenvectors, one chooses the out(−) and in(+) plane wave solutions of the Lippmann–Schwinger equation, respectively. The labels (−) and (+) indicate purely outgoing and incoming boundary conditions, respectively.

In the heuristic formulation using the Lippmann–Schwinger equations [15], the precise mathematical meaning of the out- and in-plane wave solutions is usually not stated [16]. It is understood that they are to provide a means to distinguish between in-states ϕ^+ prepared in the past, and out-states ψ^- registered by the detector in the future after they have passed the interaction region. Such a distinction is meaningless in the Hilbert space where only time symmetric solutions of the Schrödinger or Heisenberg equation—given by the unitary time evolution group—are allowed. Thus there is a contradiction between the Hilbert space axiom of quantum theory and the distinction between in- and out-states in scattering theory. Since the solutions to the Lippmann–Schwinger equation are kets $|E, j, \eta^{\mp}\rangle$, they are not elements of the Hilbert space, and one can choose them to be two solutions of the same eigenvalue equation with two different, time asymmetric, boundary conditions. This is precisely what we intend to do [19]. We replace the axiom of orthodox von Neumann quantum mechanics, which asserts that the set of prepared in-states and the set of detected observables (or out-states) are both equivalent to the Hilbert space

$$\{\text{set of prepared in-states } \phi\} = \{\text{set of prepared in-states } \psi\} = \mathcal{H} \tag{2.4}$$

by a new axiom.

This axiom states that the prepared states, defined by the preparation apparatus (accelerator), are described by

$$\{\phi^+\} = \Phi_- \subset \mathcal{H} \subset \Phi_-^\times \tag{2.5}$$

and the registered observables, defined by the registration apparatus (detector), are described by

$$\{\psi^-\} = \Phi_+ \subset \mathcal{H} \subset \Phi_+^\times \tag{2.6}$$

where $\mathcal{H}$ in (2.5) and (2.6) denotes the same Hilbert space but Φ_- and Φ_+ are Hardy spaces which are dense in $\mathcal{H}$ (see appendix (A.14), (A.15)). For the non-relativistic case this axiom is a formulation of time asymmetric boundary conditions for the solutions of the time symmetric Schrödinger and the Heisenberg differential equations, respectively. It is correct, as stated in section 3.2 of [12], that in-states ϕ^+ and out-states ψ^- do not inhabit two different Hilbert spaces. However, in contrast to what is implied in [12], the new hypothesis (2.5), (2.6) postulates that the in- and out-kets, which are generalized eigenvectors and not in $\mathcal{H}$, are from two different spaces $\Phi_\pm^\times$, because (2.5), (2.6) postulate that the sets of in-states $\{\phi^+\}$ and out-states $\{\psi^-\}$ are *different* (dense) subspaces of the same Hilbert space $\mathcal{H}$. These two dense subspaces are Hardy spaces Φ_- and Φ_+, whose wavefunctions have different but complementary analyticity properties. We shall make use of this analyticity property below, when we analytically continue the S-matrix from the real axis to the complex pole position of the resonance. We shall also make use of the Hardy space property for many other derivations.

Since a resonance appears in one particular partial wave with a definite value of angular momentum (and parity and other discrete quantum numbers which are all included in the one label j), we consider in the sum over all j in (2.3) only the angular momentum of the resonance $j = j_R$. This means we choose from the space of out-observable vectors ψ^- only the subspace with $j = j_R$. In this subspace the discrete label j of the basis vectors $|[j, \mathsf{s}], \hat{\mathbf{p}}, j_3^-\rangle$ has the fixed value j_R (which we continue to denote by j for simplicity of notation).

Every vector $\psi_{j_R}^- \equiv \psi_j^- \in \Phi_+$ has the basis vector expansion

$$\psi_j^- = \int \mathrm{d}\mathsf{s} \sum_{j_3} \int \frac{\mathrm{d}^3\hat{\mathbf{p}}}{2\hat{\mathbf{p}}^0} |[j, \mathsf{s}], \hat{\mathbf{p}}, j_3^-\rangle \psi^-(\mathsf{s}, \hat{\mathbf{p}}, j_3) \tag{2.7}$$

where the function $\psi^-(\mathsf{s}, \hat{\mathbf{p}})$ is by the new hypothesis (2.5), (2.6) Hardy functions of the variable s, analytic in the upper half plane, $\psi^-(\mathsf{s}, \hat{\mathbf{p}}) \in \Phi_+$, cf appendix.

A similar basis vector expansion holds for the vector $\phi^+ \in \Phi_-$

$$\phi^+ = \int \mathrm{d}\mathsf{s} \sum_{j_3} \int \frac{\mathrm{d}^3\hat{\mathbf{p}}}{2\hat{\mathbf{p}}^0} |[j, \mathsf{s}], \hat{\mathbf{p}}, j_3^-\rangle \phi^+(\mathsf{s}, \hat{\mathbf{p}}, j_3) \tag{2.8}$$

where $\phi^+(\mathsf{s}) \in \Phi_-$ (analytic in the lower half complex s-plane). The Dirac basis vector expansions (2.7), (2.8) can be proved as nuclear spectral resolution if the spaces $\Phi_\pm$ are nuclear spaces, which they are [18].

After these preparations, we now construct generalized vectors in terms of the Lippmann–Schwinger kets. We call them relativistic Gamow vectors. The relativistic Gamow vector is defined as the superposition of the Lippmann–Schwinger–Dirac kets $|[j, \mathsf{s}], \hat{\mathbf{p}}, j_3^-\rangle$ with the 'exact relativistic Breit–Wigner' (1.4) as the wavefunction (1.4):

$$|[j, \mathsf{s}_R], \hat{\mathbf{p}}, j_3^-\rangle = \frac{\mathrm{i}}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}\mathsf{s} |[j, \mathsf{s}], \hat{\mathbf{p}}, j_3^-\rangle \frac{1}{\mathsf{s} - \mathsf{s}_R}. \tag{2.9}$$

In contrast to the integration boundaries $(m_3 + m_4)^2 \leqslant \mathsf{s} < +\infty$ in the Dirac basis vector expansion (2.7) for the vectors $\psi_j^- \in \Phi_+$, the integration in (2.9) extends from $-\infty < \mathsf{s} < +\infty$. This means that the $|[j, \mathsf{s}_R], \hat{\mathbf{p}}, j_3^-\rangle$ are generalized vectors, i.e. elements of the space $\Phi_+^\times$.

Applying the operator $P_\mu P^\mu$ to (2.9) and using the Titchmarsh theorem for Hardy functions [19], one can show that the $|[j, \mathsf{s}_R], \hat{\mathbf{p}}, j_3^-\rangle$ are generalized eigenvectors of the total invariant mass square operator $P_\mu P^\mu$,

$$(P_\mu P^\mu)^\times |[j, \mathsf{s}_R], \hat{\mathbf{p}}, j_3^-\rangle = \mathsf{s}_R |[j, \mathsf{s}_R], \hat{\mathbf{p}}, j_3^-\rangle \tag{2.10}$$

(since the $|[j, \mathsf{s}], \hat{\mathbf{p}}, j_3^-\rangle$ under the integral of (2.9) are eigenvectors of $P_\mu P^\mu$ with eigenvalue s). They are also generalized eigenvectors of the full Hamiltonian, $P^0 = H = H_0 + V$,

$$H^\times |[j, \mathsf{s}_R], \hat{\mathbf{p}} = 0, j_3^-\rangle = \sqrt{\mathsf{s}_R} |[j, \mathsf{s}_R], \hat{\mathbf{p}} = 0, j_3^-\rangle \tag{2.11}$$

and of the momentum operators P^i,

$$P^{\times i} |[j, \mathsf{s}_R], \hat{\mathbf{p}}, j_3^-\rangle = \sqrt{\mathsf{s}_R}\, \hat{p}^i |[j, \mathsf{s}_R], \hat{\mathbf{p}}, j_3^-\rangle. \tag{2.12}$$

Thus one has an association between the 'exact' relativistic Breit–Wigner (1.4) and the space of relativistic Gamow vectors (which are the superpositions of all the kets (2.9) with the wavefunctions $\psi(\hat{\mathbf{p}})$):

$$\begin{array}{lcl} a_j^{\mathrm{BW}}(\mathsf{s}) = \dfrac{r}{s - \mathsf{s}_R} & \Longleftrightarrow & \psi^G_{[j,\mathsf{s}_R]} = \displaystyle\sum_{j_3} \int \frac{\mathrm{d}^3\hat{\mathbf{p}}}{2\hat{\mathbf{p}}^0} |[j, \mathsf{s}_R], \hat{\mathbf{p}}, j_3^-\rangle \psi_{j_3}(\hat{\mathbf{p}}) \\ -\infty < \mathsf{s} < +\infty & & \text{for all} \quad \psi(\hat{\mathbf{p}}) \in \mathcal{S}(\mathbb{R}^3) \qquad -j \leqslant j_3 \leqslant j. \end{array} \tag{2.13}$$

The space of all vectors $\psi^G_{[j,\mathsf{s}_R]}$, for all $\psi(\hat{\mathbf{p}}) \in \mathcal{S}(\mathbb{R}^3)$, $-j \leqslant j_3 \leqslant j$, i.e. the space spanned by the Gamow kets (2.9)

$$\left\{\psi^G_{[j,\mathsf{s}_R]}\right\} = \Phi_+^\times([j, \mathsf{s}_R]) \tag{2.14}$$

is a representation space of an irreducible representation $[j, \mathsf{s}_R]$ of the Poincaré *semigroup* $\mathcal{P}_+$. Thus (2.13) associates with the relativistic Breit–Wigner, the irreducible representation space of the Poincaré semigroup. This semigroup consists of all proper orthochronous Lorentz transformations and of spacetime translations into the forward light cone

$$\mathcal{P}_+ = \{(\Lambda, x) | \Lambda \in \overline{SO(3,1)},\ \det\Lambda = +1, \Lambda^0_0 \geqslant 1, x^2 = t^2 - \mathbf{x}^2 \geqslant 0, t \geqslant 0\}. \tag{2.15}$$

These causal (forward) Poincaré semigroup representations are characterized by

1. spin (parity) j given by the jth partial wave amplitude in which the resonance occurs

$$a_j(\mathsf{s}) = a_j^{\mathrm{BW}}(\mathsf{s}) + B_j(\mathsf{s}). \tag{2.16}$$

2. the complex mass squared s_R (with $\mathrm{Im}\, \mathsf{s}_R < 0$) given by the pole position of $a_j^{\mathrm{BW}}(\mathsf{s})$ of (1.4).
3. 'minimally complex momentum', $p = \sqrt{\mathsf{s}_R}\, \hat{p}$ where $\hat{p}$ is real.

The restriction to representations of the Poincaré transformations with 'minimally complex' momentum is necessary, because we need to assure that the four-velocity $\hat{p}$ is real, since the boost (rotation-free Lorentz transformation from rest to the four-velocity $\hat{p}$ or to three-velocity $\mathbf{v} = \hat{\mathbf{p}}/\gamma$, $\gamma = 1/\sqrt{1 - \mathbf{v}^2}$) is a function of the real parameter $\hat{p}$. The third condition then assures that the restriction of the representation $[j, \mathsf{s}_R]$ to the homogeneous Lorentz subgroup is the same unitary representation as occurs in Wigner's unitary representation for stable particles $[j, m^2]$.

The representation of the semigroup (2.15) and the action of the operator $U^\times(\Lambda, x)$ in the space $\Phi_+^\times([j, \mathsf{s}_R])$ can only be appreciated in comparison with the unitary representation operator $U^\dagger(\Lambda, x)$ of the Poincaré group

$$\mathcal{P} = \left\{(\Lambda, x) | \Lambda \in \overline{SO(3,1)}, \det\Lambda = +1, \Lambda^0_0 \geqslant 1\right\}. \tag{2.17}$$

For the unitary operator $U^\dagger_{[j,m^2]}(\Lambda, x) = U^\dagger(\Lambda, x) = U((\Lambda, x)^{-1}) = U(\Lambda^{-1}, -\Lambda^{-1}x)$ in the irreducible representation space $\mathcal{H}(j, m^2)$, the action of the operators on the momentum basis vectors is written as [13]

$$U^\dagger(\Lambda, x)|\hat{\mathbf{p}}, j_3\rangle = \mathrm{e}^{-\mathrm{i}p\cdot x} \sum_{j_3'} D^j_{j_3 j_3'}(W(\Lambda^{-1}, \hat{p}))|\Lambda^{-1}\hat{\mathbf{p}}, j_3'\rangle \qquad -\infty < t < \infty \tag{2.18}$$

where $\mathrm{e}^{-\mathrm{i}px} = \mathrm{e}^{-\mathrm{i}\gamma m(t-\mathbf{v}\cdot\mathbf{x})}$ and $W(\Lambda^{-1}, \hat{p}) = L^{-1}(\Lambda^{-1}\hat{p})\Lambda^{-1}L(\hat{p})$ is the Wigner rotation and $L(\hat{p})$ is the boost

$$L^\mu{}_\nu(\hat{p}) = \begin{pmatrix} \frac{p^0}{m} & -\frac{p_n}{m} \\ \frac{p^m}{m} & \delta^m_n - \frac{\frac{p^m}{m}\frac{p_n}{m}}{1+\frac{p^0}{m}} \end{pmatrix} \tag{2.19}$$

which has the property

$$L^{-1}(\hat{p})^\mu{}_\nu p^\nu = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.20}$$

The boost $L(\hat{p})$ and therewith $W(\Lambda, \hat{p})$ depends upon $\hat{p}$ and not upon the momentum $p = \sqrt{\mathsf{s}}\hat{p}$. It is this property that allows us to construct the minimally complex representations $[j, \mathsf{s}_R]$ by analytic continuation of the Lippmann–Schwinger kets with real energy s to the Gamow kets with complex energy s_R

$$|[j, s], \hat{\mathbf{p}}, j_3^-\rangle \rightarrow |[j, \mathsf{s}_R], \hat{\mathbf{p}}, j_3^-\rangle. \tag{2.21}$$

We perform this analytic continuation to complex values of s in such a way that $\hat{p}$ remains unaffected; $\hat{p}$ = real. The momenta then become 'minimally' complex, meaning that the momentum p is given as the product of the complex invariant mass $\sqrt{\mathsf{s}}$ with the real four-velocity vector $\hat{p}$ ($\hat{p}_\mu\hat{p}^\mu = 1$), $p = \sqrt{\mathsf{s}}\hat{p}$. Since the velocity basis vectors differ from the momentum eigenvectors by a trivial normalization factor $N(m)$ which depends upon the measure $\mathrm{d}\mu(\hat{p}) = \mathrm{d}^3\mathbf{p}/2\mathbf{p}^0$ (or equivalently upon the δ-normalization)

$$|[j, m^2], \hat{\mathbf{p}}, j_3\rangle = N(m)|[j, m^2], \mathbf{p}, j_3\rangle \in \Phi^\times \supset \mathcal{H}(m^2, j) \tag{2.22}$$

one can substitute in the transformation formulae $[j, m^2]$, the four-velocity kets (2.22) for the momentum kets as already done in (2.18).

However, the usual non-rigorous notation (2.18) is not correct for either basis vectors since these kets are not in Hilbert space. In order to make (2.18) mathematically rigorous we have to replace the $U^\dagger$ which acts in $\mathcal{H}(m^2, j)$, by its unique extension $U^\times(\Lambda, x) \supset U^\dagger(\Lambda, x)$ in $\Phi^\times \supset \mathcal{H}(m^2, j)$.

If Φ is the space of differentiable vectors of $\mathcal{H}(m^2, j)$ endowed with a topology defined by the Nelson operator [20, 21], then $U^\times(\Lambda, x)$ is also a representation of the Poincaré group [21]. The same would hold for the reducible representations if the functions $\psi_j(\mathsf{s})$ in (2.3) were Schwartz space functions. If, however, the $\psi_j(\mathsf{s})$ are Hardy functions and the basis vectors are the Lippmann–Schwinger scattering states defined as functionals $|[j, \mathsf{s}], \hat{\mathbf{p}}, j_3^-\rangle \in \Phi_+^\times$, then their transformation properties are the same as for the $|[j, \mathsf{s}], \hat{\mathbf{p}}, j_3\rangle \in \Phi^\times$ except that the $U_+^\times(\Lambda, x) \in \Phi_+^\times$ are only defined for the semigroup[6] $\mathcal{P}_+$. The transformation property of the Lippmann–Schwinger kets under the transformations $(\Lambda, x) \in \mathcal{P}_+$ is given in [22]

$$U^\times(\Lambda, x)|[j, s], \hat{\mathbf{p}}, j_3^-\rangle = \mathrm{e}^{-\mathrm{i}p\cdot x} \sum_{j_3'} D^j_{j_3' j_3}(W(\Lambda^{-1}, \hat{p}))|[j, \mathsf{s}], \Lambda^{-1}\hat{\mathbf{p}}, j'^-_3\rangle$$

$$\text{only for} \quad x^2 \geqslant 0 \quad \text{and} \quad t \geqslant 0. \tag{2.23}$$

[6] The reason for this is the (small and usually not mentioned) negative imaginary part of $s(p^0)$, $\sqrt{\mathsf{s}} = \sqrt{\mathsf{s}_0} - \mathrm{i}\epsilon$ for the Lippmann–Schwinger kets.

The transformation of the Gamow kets $|[j, \mathsf{s}_R], \hat{\mathbf{p}}, j_3^-\rangle$ under $(\Lambda, x) \in \mathcal{P}_+$ is very similar to (2.23) and given by [23]

$$U^\times(\Lambda, x)|[j, \mathsf{s}_R], \hat{\mathbf{p}}, j_3^-\rangle = \mathrm{e}^{-\mathrm{i}\gamma\sqrt{\mathsf{s}_R}(t-\mathbf{x}\cdot\mathbf{v})} \sum_{j_3'} D^j_{j_3' j_3}(W(\Lambda^{-1}, \hat{p}))|[j, \mathsf{s}_R], \Lambda^{-1}\hat{\mathbf{p}}, j_3'^-\rangle$$
$$\text{only for} \quad x^2 \geqslant 0 \quad \text{and} \quad t \geqslant 0 \tag{2.24}$$

where $\mathbf{v}$ is the three-velocity, and $\hat{\mathbf{p}} = \gamma\mathbf{v}$, where $\gamma = 1/\sqrt{1-\mathbf{v}^2} = \sqrt{1+\hat{\mathbf{p}}^2} = \hat{p}^0$.

For a spacetime translation of the out-observable, $|\psi_\eta^-\rangle\langle\psi_\eta^-|$ (described by any $\psi_\eta^- \in \Phi_+$) relative to the decaying state (described by the Gamow state $|[\mathsf{s}_R, j], \hat{\mathbf{p}}, j_3^-\rangle$) we obtain the Born probability amplitude from (2.24)

$$\begin{aligned}
\langle U(I, x)\psi_\eta^-|[\mathsf{s}_R, j], \hat{\mathbf{p}}, j_3^-\rangle &= \langle\psi_\eta^-|U^\times(I, x)|[\mathsf{s}_R, j], \hat{\mathbf{p}}, j_3^-\rangle \\
&= \langle\psi_\eta^-|\mathrm{e}^{-\mathrm{i}x^\mu P_\mu^\times}|[\mathsf{s}_R, j], \hat{\mathbf{p}}, j_3^-\rangle \\
&= \langle\psi_\eta^-|\mathrm{e}^{-\mathrm{i}[H^\times t - \mathbf{x}\cdot\mathbf{P}^\times]}|[\mathsf{s}_R, j], \hat{\mathbf{p}}, j_3^-\rangle \\
&= \mathrm{e}^{-\mathrm{i}\gamma\sqrt{\mathsf{s}_R}(t-\mathbf{x}\cdot\mathbf{v})}\langle\psi_\eta^-|[\mathsf{s}_R, j], \hat{\mathbf{p}}, j_3^-\rangle \\
&= \mathrm{e}^{-\frac{\Gamma_R}{2}\gamma(t-\mathbf{x}\cdot\mathbf{v})}\mathrm{e}^{-\mathrm{i}\gamma M_R(t-\mathbf{x}\cdot\mathbf{v})}\langle\psi_\eta^-|[\mathsf{s}_R, j], \hat{\mathbf{p}}, j_3^-\rangle \\
&\qquad \text{for} \quad x^2 \geqslant 0 \quad t \geqslant 0 \text{ only.}
\end{aligned} \tag{2.25}$$

The absolute value square of (2.25) is the Born probability (density) to find the decay products η in the spacetime-translated Gamow state $\psi^G_{\mathsf{s}_R}(x) = \mathrm{e}^{-\mathrm{i}x^\mu P_\mu^\times}|[\mathsf{s}_R, j], \hat{\mathbf{p}}, j_3^-\rangle$. We have thus obtained for the decay rate the exponential decay law with time dilation if the relative velocity $\mathbf{v}$ of the detector and the decaying state are different from zero. A detailed discussion of how one measures this will be given in [22]. Here we want to consider the special case of the time evolution in the rest frame of the decaying state, $\mathbf{v} = 0$, for which we obtain from (2.25)

$$\begin{aligned}
\left|\psi^G_{\mathsf{s}_R}(t)^-\right\rangle &\equiv \mathrm{e}^{-\mathrm{i}H^\times t}|[j, \mathsf{s}_R], \hat{\mathbf{p}} = 0, j_3^-\rangle = \mathrm{e}^{-\mathrm{i}\sqrt{\mathsf{s}_R}t}|[j, \mathsf{s}_R], \hat{\mathbf{p}} = 0, j_3^-\rangle \\
&= \mathrm{e}^{-\mathrm{i}M_R t}\,\mathrm{e}^{-\Gamma_R t/2}|[j, \mathsf{s}_R], \hat{\mathbf{p}} = 0, j_3^-\rangle \qquad \text{for} \quad t \geqslant 0 \quad \text{only.}
\end{aligned} \tag{2.26}$$

We now apply (2.26) to the quasistationary state of the Z-boson created in the formation process (1.1). We choose $\psi^G_{[j,\mathsf{s}_R]} = |Z\rangle$, using the parametrization $\mathsf{s}_R = \left(M_R - \mathrm{i}\frac{\Gamma_R}{2}\right)^2$ from (1.6) and we choose for the out-observable (i.e. the vectors representing the decay products of the resonance) $|\psi_\eta^-\rangle = |f\bar{f}\rangle$. From (2.26) we then obtain that the probability rate for transitions of the relativistic Gamow state into any decay product η decreases by the exponential law:

$$\left|\left\langle\psi_\eta^-\psi^G_{[j,\mathsf{s}_R]}(t)^-\right\rangle\right|^2 = \mathrm{e}^{-\Gamma_R t}\left|\left\langle\psi_\eta^-[j, \mathsf{s}_R], \hat{\mathbf{p}} = 0, j_3^-\right\rangle\right|^2 \tag{2.27}$$

with Γ_R defined as the width of the Breit–Wigner energy wavefunction $\Gamma_R = -2\,\mathrm{Im}\sqrt{\mathsf{s}_R}$ in (2.9).

Initially and without the transformation property (2.25), (2.26) of the Poincaré semigroup representation for the Gamow vector (2.9), the width Γ_R had nothing to do with time evolution. It was just the width (one of many (1.5), (1.6), ...) of the resonance scattering amplitude (1.4) which was used as the energy distribution of the Gamow vector (2.9).

From the transformation formula (2.24) and therewith the exponential time evolution (2.27), it follows that the lifetime of the quasistable particle represented by the relativistic Gamow vector (2.9), is given by $\tau = \hbar/\Gamma_R$ with width $\Gamma_R = -2\,\mathrm{Im}\sqrt{\mathsf{s}_R}$. Only this width Γ_R, and not any of the other possible widths, Γ_Z, $\bar{\Gamma}_Z$, etc has this property. For the Gamow state with width Γ_R we have thus predicted by (2.27) that the probability rate for any decay products η fulfils the exponential law with the lifetime (in the rest frame) of the relativistic resonance given as

$$\tau_R = \hbar/\Gamma_R. \tag{2.28}$$

In addition the Gamow vector predicts time asymmetry and distinguishes a time direction which is not possible with unitary representations of the Poincaré group in the Hilbert space. The restriction $t \geqslant t_0 \equiv 0$ correctly describes the physical situation because the decay products η, described by ψ_η^-, can be detected only after the decaying state R, described by the Gamow vector $\psi_{[j,\mathsf{s}_R]}(t)$, has been created at $t = t_0 \equiv 0$.

Returning to the practical problem of the resonance mass and width of the Z-boson mentioned in section 1, we conclude from (2.25)–(2.27) that if (2.28) is to hold for relativistic resonances, then the width of the Z-boson must be given by the parametrization (1.6) for the lineshape (1.4). The real resonance mass is then the M_R in the exponential of the phase factor of (2.26), i.e. $M_R = \mathrm{Re}\sqrt{\mathsf{s}_R}$ and not the peak position $\bar{M}_Z = M_R\sqrt{1 - 1/4(\Gamma_R/M_R)^2}$ of the relativistic Breit–Wigner or any other parametrization of the lineshape used and reported in the review of particle properties [2]. Determining the values Γ_R, M_R from a fit of (1.4) to the experimental lineshape data one obtains the experimental value of the Z-boson mass as [3]

$$M_R = \mathrm{Re}\sqrt{\mathsf{s}_R} = 91.1626 \pm 0.0031\,\mathrm{GeV}. \tag{2.29}$$

From the exponential time evolution derived for the relativistic Gamow vector it follows that the 'width' of the relativistic Breit–Wigner Γ_R (not $\bar{\Gamma}_Z$, Γ_Z or any other Γ) is the inverse lifetime $\tau_R = \hbar/\Gamma_R$. From the semigroup character of the representation $[j, \mathsf{s}_R]$ follows the time asymmetry (microphysical irreversibility).

3. The exponentially evolving Gamow vector from each S-matrix pole

The definition of a resonance in section 2 by an irreducible representation $[j, \mathsf{s}_R = (M_R - \mathrm{i}\Gamma_R/2)^2]$ of the Poincaré semigroup was empirically suggested by the experimental lineshape of a relativistic resonance (in particular the Z-boson lineshape in section 2, and also by other relativistic resonances [10]) and theoretically supported by the perturbation theoretical treatment in various renormalization schemes [6–9]. The field theoretical arguments, using gauge independence for the renormalization scheme, favoured the relativistic Breit–Wigner amplitude $a_j^{\mathrm{BW}}(\mathsf{s})$ of (1.4) as the partial wave amplitude for the resonance *per se*. This relativistic Breit–Wigner amplitude was then used to define a relativistic Gamow vector in terms of the out-plane wave solutions of the Lippmann–Schwinger equation by a formula (2.9) which is essentially the Titchmarsh theorem for Hardy functions (when taken at $\psi^- \in \Phi_+$).

The most widely accepted definition of resonances in non-relativistic as well as relativistic physics, is the definition as a first-order pole in the second (or higher) sheet of the analytically continued S-matrix at pole positions $\mathsf{s} = \mathsf{s}_{R_i}$ in the lower half plane. We shall use this definition to establish a correspondence between a relativistic Breit–Wigner for each S-matrix pole s_{R_i} and an irreducible representation space $[j, \mathsf{s}_{R_i}]$ for Gamow vectors. This will associate with the background amplitude, $B_j(\mathsf{s})$ in (1.2), a background vector ϕ^{bg} whose time evolution is non-exponential.

We shall now derive the relativistic Gamow vector from the resonance pole of the relativistic S-matrix at $\mathsf{s} = \mathsf{s}_R$. A relativistic resonance defined by a second sheet pole of the S-matrix at the complex value $\mathsf{s} = \mathsf{s}_R$ is thus characterized by two real parameters which we choose to call $\mathrm{Re}\,(\sqrt{\mathsf{s}_R}) \equiv M_R$ and $\mathrm{Im}\sqrt{\mathsf{s}_R} \equiv -\Gamma_R/2$ as in (1.6). The reason for this choice is relation (2.28).

We start with the S-matrix element between a prepared in-state ϕ^+ and a detected out-observable ψ^-. We assume the asymmetric boundary conditions $\phi^+ \in \mathbf{\Phi}_-$ and $\psi^- \in \mathbf{\Phi}_+$ of the new hypothesis (2.5), (2.6)

$$
\begin{aligned}
(\psi^{\text{out}}, \phi^{\text{out}}) &= (\psi^{\text{out}}, S\phi^{\text{in}}) = (\Omega^{-}\psi^{\text{out}}, \Omega^{+}\phi^{\text{in}}) = (\psi^{-}, \phi^{+}) \\
&= \sum_{j, j_3, n} \int \frac{\mathrm{d}^3\hat{\mathbf{p}}}{2\hat{E}}\, \mathrm{d}\mathsf{s} \sum_{j', j_3', n'} \frac{\mathrm{d}^3\hat{\mathbf{p}}'}{2\hat{E}'}\, \mathrm{d}\mathsf{s}' \langle \psi^{-} | [\mathsf{s}, j], n, j_3, \hat{\mathbf{p}}^{-} \rangle \\
&\quad \times \langle \hat{\mathbf{p}}, j_3, [\mathsf{s}, j], n | S | [j', \mathsf{s}'], j_3', \hat{\mathbf{p}}', n' \rangle \langle^{+} j_3', \hat{\mathbf{p}}', [j', \mathsf{s}'], n' | \phi^{+} \rangle.
\end{aligned}
\tag{3.1}
$$

In this S-matrix element, ϕ^{in} describes the asymptotically-free in-state that is prepared, e.g. by the accelerator, outside the interaction region. This ϕ^{in} becomes the ϕ^{+} in the interaction region (of the two beams in $e^{+}e^{-} \to Z \to \bar{f} f$), the energy distribution in the beams is described by the wavefunctions $\phi^{\text{in}}(\mathsf{s}) = \phi^{+}(\mathsf{s})$. The out-state vector ψ^{out} describes the detected out-particles (e.g., a particular $\bar{f} f$) when they are asymptotically free. It comes from the ψ^{-} in the interaction region and its wavefunction $\psi^{\text{out}}(\mathsf{s}) = \psi^{-}(\mathsf{s})$ describes the energy resolution of the detectors. ψ^{-} is defined by the registration apparatus (detector)—for which reason $|\psi^{-}\rangle\langle\psi^{-}|$ should be called observable rather than out-state. The kets $|[j, \mathsf{s}], b^{\mp}\rangle$ are the generalized eigenvectors of the exact energy operator $P_\mu P^\mu$ as stated in equation (2.10).

The kets $|[j, \mathsf{s}], b\rangle$ are the corresponding eigenvectors of the asymptotically-free energy operator and

$$
\psi^{\text{out}}(\mathsf{s}) \equiv \langle b, [j, \mathsf{s}] | \psi^{\text{out}} \rangle = \langle^{-} b, [j, \mathsf{s}] | \psi^{-} \rangle \equiv \psi^{-}(\mathsf{s}) \tag{3.2}
$$

$$
\phi^{\text{in}}(\mathsf{s}) \equiv \langle b, [j, \mathsf{s}] | \phi^{\text{in}} \rangle = \langle^{+} b, [j, \mathsf{s}] | \phi^{+} \rangle \equiv \phi^{+}(\mathsf{s}). \tag{3.3}
$$

In addition to the property that $|\psi^{\text{out}}(\mathsf{s})|^2 = |\psi^{-}(\mathsf{s})|^2$ and $|\phi^{\text{in}}(\mathsf{s})|^2 = |\phi^{+}(\mathsf{s})|^2$ be a smooth function of s (since they describe the apparatus resolution), we also require according to our new hypothesis (2.5), (2.6) that these functions have certain analyticity properties; precisely they are Hardy functions[7]. In expressing the matrix element (ψ^{-}, ϕ^{+}) by the rhs of (3.1), we have used for ψ^{-} and ϕ^{+} the basis vector expansions (2.7) and (2.8). We explicitly included the additional quantum number n (e.g. channel or species label) in (3.1). For the Lorentz invariant integration we choose $\mathrm{d}\mu(b) = \mathrm{d}^3\hat{\mathbf{p}}/(2\hat{p}^0)$. And we have written the S-matrix as

$$
\begin{aligned}
\langle^{-} \hat{\mathbf{p}}, j_3, [\mathsf{s}, j], n | \hat{\mathbf{p}}', j_3', [\mathsf{s}', j'], n'^{+} \rangle &= (\Omega^{-} | \hat{\mathbf{p}}, j_3, [\mathsf{s}, j], n \rangle, \Omega^{+} | \hat{\mathbf{p}}', j_3', [\mathsf{s}', j'], n' \rangle) \\
&= \langle \hat{\mathbf{p}}, j_3, [\mathsf{s}, j], n | \Omega^{-\dagger}\Omega^{+} | \hat{\mathbf{p}}', j_3', [\mathsf{s}', j'], n' \rangle \\
&= \langle \hat{\mathbf{p}}, j_3, [\mathsf{s}, j], n | S | \hat{\mathbf{p}}', j_3', [\mathsf{s}', j'], n' \rangle.
\end{aligned}
\tag{3.4}
$$

This enormous S-matrix can be considerably reduced using symmetry properties. From the invariance of the S-operator with respect to Poincaré transformations one can show that the S-matrix element (3.4) can be written as

$$
\langle \hat{\mathbf{p}}, j_3, [\mathsf{s}, j], n | S | \hat{\mathbf{p}}', j_3', [\mathsf{s}', j'], n' \rangle = 2\hat{E}(\hat{p})\delta^3(\hat{p} - \hat{p}')\delta(\mathsf{s} - \mathsf{s}')\delta_{j_3 j_3'}\delta_{jj'} \langle n \| S_j(\mathsf{s}) \| n' \rangle \tag{3.5}
$$

where $\langle n \| S_j(\mathsf{s}) \| n' \rangle$ is the reduced S-matrix element which depends upon j which labels the partial wave, and the particle species and channel quantum numbers n, n'. For a fixed initial state n' it is written in terms of the scattering amplitudes used in section 1:

$$
\langle n \| S_j(\mathsf{s}) \| n' \rangle = S_j^{(n)}(\mathsf{s}) = \begin{cases} 2\mathrm{i}a_j(\mathsf{s}) + 1 & \text{for elastic scattering } n = n' \\ 2\mathrm{i}a_j^{(n)}(\mathsf{s}) & \text{for reaction from } n' \text{ into the channel } n. \end{cases} \tag{3.6}
$$

[7] This is related to causality based on the fact that the in-state ϕ^{+} must be prepared first before the out-observable ψ^{-} can be detected in it [23]. The S-matrix element $|(\psi^{-}, \phi^{+})|^2$ describes the probability of detecting the observable ψ^{-} in the state ϕ^{+} (Born probability). This is also expressed by the asymptotically-free quantities $|(\psi^{\text{out}}, \phi^{\text{out}})|^2$, where $\phi^{\text{out}} = S\phi^{\text{in}}$ is a state (not an observable such as ψ^{out}) which is defined by the preparation apparatus as a ϕ^{in} and the dynamics described by the S-operator (or by the Hamiltonian H if S is calculated in terms of $H = H_0 + V$).

Here $a_j(\mathsf{s})$ is the partial wave amplitude of (1.2) for $f\bar{f} = e^-e^+$ and $a_j^{(n)}(\mathsf{s})$ are the partial wave amplitudes used in (1.2) for $e^+e^- \to Z \to \mu\bar{\mu}$, etc. We insert (3.5) and (3.6) into (3.1) and obtain for the S-matrix element (suppressing the additional quantum numbers n again)

$$(\psi^-, \phi^+) = \sum_j \int_{m_0^2}^{\infty} \mathrm{d}\mathsf{s} \sum_{j_3} \int \frac{\mathrm{d}^3\hat{\mathbf{p}}}{2\hat{E}} \langle \psi^- |[j, \mathsf{s}], j_3, \hat{\mathbf{p}}^-\rangle S_j(\mathsf{s}) \langle^+ j_3, \hat{\mathbf{p}}, [j, \mathsf{s}]|\phi^+\rangle. \tag{3.7}$$

After we use the Poincaré invariance of the S-matrix using the three-velocity basis vectors $|[j, \mathsf{s}], b^\mp\rangle = |[j, \mathsf{s}], j_3, \hat{\mathbf{p}}^\mp\rangle$ we again ignore the degeneracy quantum numbers b and we consider only the jth partial S-matrix element (where j is the spin-parity of the resonance we want to consider). With this simplified notation $|[j, s]j_3, \hat{\mathbf{p}}^\mp\rangle \equiv |s^\mp\rangle$ we write the jth term in (3.7) as

$$(\psi^-, \phi^+)_j = \int_{m_0^2}^{\infty} \mathrm{d}\mathsf{s} \langle \psi^- |\mathsf{s}^-\rangle S_j(\mathsf{s}) \langle^+ \mathsf{s}|\phi^+\rangle \tag{3.8}$$

where the energy wavefunctions (3.2) and (3.3) are according to the new hypothesis (2.5), (2.6) of Hardy functions[8] which were suggested by the Lippmann–Schwinger equation,

$$\psi^-(\mathsf{s}) = \langle^- \mathsf{s}|\psi^-\rangle = \langle^- b, [j, \mathsf{s}]|\psi^-\rangle \in \mathsf{H}_+^2 \cap S|_{\mathbb{R}_{m_0}} \tag{3.9}$$

$$\phi^+(\mathsf{s}) = \langle^+ \mathsf{s}|\phi^+\rangle = \langle^+ b, [j, \mathsf{s}]|\phi^+\rangle \in \mathsf{H}_-^2 \cap S|_{\mathbb{R}_{m_0}} \tag{3.10}$$

ϕ^+ describes the prepared in-state (e^+e^-) and $|\phi^+(\mathsf{s})|^2 = |\phi^{\mathrm{in}}(\mathsf{s})|^2$ describes the energy distribution of the beam. ψ^- describes the observed out-observable ($e^+e^-, \mu\bar{\mu}, \tau\bar{\tau}, \ldots$) which is registered by the detector, and $|\psi^-(\mathsf{s})|^2 = |\psi^{\mathrm{out}}(\mathsf{s})|^2$ describes the detector efficiency. Therefore they should be smooth, rapidly decreasing functions.

All of our following results will be a consequence of the new hypothesis of (2.5), (2.6) or its realization in terms of energy wavefunctions given in (3.9), (3.10). Except for this new hypothesis, all other assumptions which we shall use are the standard axioms of quantum theory and relativistic invariance.

Since we describe the interaction in terms of the S-operator, the dynamics is encapsulated in the property of the S-matrix, $S_j(\mathsf{s})$, as a function of the scattering energy squared s. We shall use for $S_j(\mathsf{s})$ the standard assumption of polynomial boundedness and analyticity. Since we are interested in the resonances we also make the usual first-order pole assumption [29].

In order to be as simple as possible we shall consider the specific model that there are $N = 2$ resonances in the jth partial wave, each described by a first-order pole at the position $\mathsf{s} = \mathsf{s}_{R_1}$ and $\mathsf{s} = \mathsf{s}_{R_2}$ in the second sheet. As a consequence of this and the hypothesis (3.9), (3.10), the integrand in (3.8) is analytic in the lower half plane of the second sheet except for the two poles at $\mathsf{s} = \mathsf{s}_{R_i}$. This is depicted in figure 1(a) which also shows the cut along the real axis from $m_0^2 \leqslant s < \infty$. We can deform the contour of integration in (3.8) from the positive real line on the first sheet through the cut into the lower half plane of the second sheet where the integral over the infinite semicircle has been omitted since it is zero as a consequence of (3.9), (3.10) and the boundedness property of $S_j(s)$. The result of this contour deformation is shown in figure 1(b) where the infinite semicircle in the complex plane has been omitted. The rhs of (3.8) becomes

$$(\psi^-, \phi^+) = \int_{m_0^2}^{-\infty_{II}} \mathrm{d}\mathsf{s} \langle \psi^- |\mathsf{s}^-\rangle S_{II}(\mathsf{s}) \langle^+ \mathsf{s}|\phi^+\rangle + \oint_{C_1} \mathrm{d}\mathsf{s} \langle \psi^- |\mathsf{s}^-\rangle S_{II}(\mathsf{s}) \langle^+ \mathsf{s}|\phi^+\rangle + \oint_{C_2} \mathrm{d}\mathsf{s} \langle \psi^- |\mathsf{s}^-\rangle S_{II}(\mathsf{s}) \langle^+ \mathsf{s}|\phi^+\rangle. \tag{3.11}$$

[8] The function spaces $\mathsf{H}_\pm \cap S|_{\mathbb{R}_{m_0}}$ are realizations of the abstract Hardy spaces $\Phi_\pm$; in this way (3.9), (3.10) are equivalent to (2.5), (2.6). See also the appendix on rigged Hilbert spaces at the end of this paper.

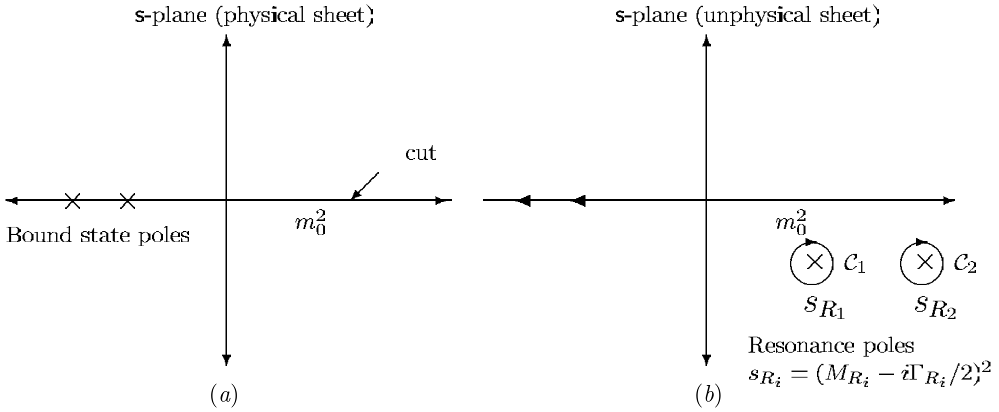

Figure 1. The two sheeted S-matrix. The jth partial S-matrix $S_j(E)$ is an analytic function on a Riemann energy surface cut along the positive real axis from $m_0^2 \leqslant \mathsf{s} < \infty$ indicated in (a). The integration in (3.8) is along the cut in (a), either on the lower edge of the 'physical sheet' or along the upper edge of the second sheet. The contour of integration can be deformed into the lower half plane of the second sheet, and ultimately into the contours around the two resonance poles indicated by $\times$ and into an integral from m_0^2 to $-\infty$ along the upper edge of the second sheet. This is shown in (b); the arrows indicate the direction of integration. Thus we have the equality of the integrals in (3.8) and (3.11).

Here C_i is the circle around the pole at s_{R_i}, and the first integral extends along the negative real axis in the second sheet (indicated by $-\infty_{II}$). The first term has nothing to do with any of the resonances, it is the non-resonant background term,

$$\int_{m_0^2}^{-\infty_{II}} \mathrm{d}\mathsf{s}\langle\psi^-|\mathsf{s}^-\rangle S_{II}(\mathsf{s})\langle^+\mathsf{s}|\phi^+\rangle \equiv \langle\psi^-|\phi^{bg}\rangle \tag{3.12}$$

which we express as the matrix element of ψ^- with a generalized vector ϕ^{bg} that is defined by it

$$\phi^{bg} \equiv \int_{m_0^2}^{-\infty} \mathrm{d}s|s^-\rangle\langle^+s|\phi^+\rangle S_{II}(s). \tag{3.13}$$

It will be discussed further in section 4.

We now consider each integral along C_i around each pole at s_{R_i} separately. For each integral we use the expansion around the pole s_{R_i} separately

$$S(\mathsf{s}) = \frac{R^{(i)}}{\mathsf{s} - \mathsf{s}_{R_i}} + R_0 + R_1\left(\mathsf{s} - \mathsf{s}_{R_i}\right) + \cdots. \tag{3.14}$$

For each of the two (or N) integrals *separately* we evaluate the integrals around each pole s_{R_i}. Then we obtain for each of these pole terms the following results:

$$(\psi^-, \phi^+)_{\text{pole term}_i} = \oint_{\hookleftarrow C_i} \mathrm{d}\mathsf{s}\langle\psi^-|\mathsf{s}^-\rangle S(\mathsf{s})\langle^+\mathsf{s}|\phi^+\rangle \tag{3.15}$$

$$= \oint_{\hookleftarrow C_i} \mathrm{d}\mathsf{s}\langle\psi^-|\mathsf{s}^-\rangle \frac{R^{(i)}}{\mathsf{s} - \mathsf{s}_{R_i}}\langle^+\mathsf{s}|\phi^+\rangle \tag{3.16}$$

$$= -2\pi\mathrm{i}R^{(i)}\left\langle\psi^-\middle|\mathsf{s}_{R_i}^-\right\rangle\left\langle^+\mathsf{s}_{R_i}\middle|\phi^+\right\rangle \tag{3.17}$$

$$= \int_{-\infty_{II}}^{\infty} \mathrm{d}\mathsf{s}\langle\psi^-|\mathsf{s}^-\rangle\langle^+\mathsf{s}|\phi^+\rangle\frac{R^{(i)}}{\mathsf{s} - \mathsf{s}_{R_i}}. \tag{3.18}$$

To get from (3.15) to (3.16) we used (3.20). To get from (3.16) to (3.17), the Cauchy theorem has been applied; to get from (3.16) to (3.18), the contour C_i of each integral separately has been deformed into the integral along the real axis from $-\infty_{II} < \mathsf{s} < +\infty$ (and an integral along the infinite semicircle, which vanishes because of the Hardy class property). The equality between (3.17) and (3.18) is also called the Titchmarsh theorem for Hardy class functions. The integral (3.18) extends from $\mathsf{s} = -\infty_{II}$ in the second sheet along the real axis to $\mathsf{s} = 0$ and then from $\mathsf{s} = 0$ to $\mathsf{s} = +\infty$ in either sheet. (It does not matter whether we take the second part of the integral over the physical values of s, $m_0^2 \leqslant \mathsf{s} < \infty$, immediately below the real axis in the second sheet or in the first sheet immediately above the real axis). The major contribution to the integral comes from the physical values $m_0^2 \leqslant \mathsf{s} < \infty$, if s_{R_i} is not too far from the real axis.

The integral in (3.18) contains the Breit–Wigner amplitude

$$a_j^{\mathrm{BW}_i} = \frac{R^{(i)}}{\mathsf{s} - \mathsf{s}_{R_i}} \qquad \text{but with } -\infty_{II} < \mathsf{s} < +\infty \quad i = 1, 2, \ldots, N. \tag{3.19}$$

Unlike the conventional Breit–Wigner for which s is taken (if one worries about these mathematical details) over $m_0^2 \leqslant \mathsf{s} < +\infty$, the Breit–Wigner (3.19) is an idealized or exact Breit–Wigner whose domain extends to $-\infty_{II}$ in the second (unphysical) sheet.

By (3.18) we have associated each resonance at s_{R_i} with an exact Breit–Wigner (3.19) which we obtain by omitting the integral over the arbitrary function $\overline{\langle^-\mathsf{s}|\psi^-\rangle}\langle^+\mathsf{s}|\phi^+\rangle \in \Phi_-$ from (3.18). By (3.17) we have associated each resonance at s_{R_i} with vectors $\left|\mathsf{s}_{R_i}^-\right\rangle = \left|\left[j, \mathsf{s}_{R_i}\right], b^-\right\rangle$ which we call Gamow vectors.

We obtain a representation of the Gamow vectors by using the equality between (3.17) and (3.18) and omitting the arbitrary $\psi^- \in \Phi_+$ (which represents the decay products defined by the detector). For this defining relation of the relativistic Gamow vectors we shall use the notation that includes the degeneracy quantum numbers b. Thus, considered as a functional equation in $\Phi_+^\times$, we obtain

$$\begin{aligned}\left|\left[j, \mathsf{s}_{R_i}\right], b^-\right\rangle &= \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\mathsf{s}|[j, \mathsf{s}], b^-\rangle \frac{1}{\mathsf{s} - \mathsf{s}_{R_i}} \frac{\langle^+\mathsf{s}|\phi^+\rangle}{\left\langle^+\mathsf{s}_{R_i}\middle|\phi^+\right\rangle} \\ &= \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\mathsf{s}|[j, \mathsf{s}], b^-\rangle \frac{1}{\mathsf{s} - \mathsf{s}_{R_i}} \qquad i = 1, 2, \ldots, N.\end{aligned} \tag{3.20}$$

This is the same formula by which we defined the relativistic Gamow kets in (2.9) in terms of a relativistic Breit–Wigner wavefunction. Here we derived it from the S-matrix pole. The Gamow kets (3.20) are a superposition of the exact—not asymptotically free[9]—'out states' $|[j, \mathsf{s}], b^-\rangle$.

The degeneracy quantum numbers b of the Gamow kets $\left|\left[j, \mathsf{s}_{R_i}\right], b^-\right\rangle$ are the same as those chosen for the Dirac–Lippmann–Schwinger kets $|[j, \mathsf{s}], b^-\rangle$. However, whereas for the Dirac–Lippmann–Schwinger kets one can choose for $b = b_1, \ldots, b_n$ the eigenvectors of any complete set of observables, one does not have the same freedom for the b in the Gamow kets, since in the contour deformations that one uses to get from (3.8) to (3.11) and ultimately to (3.15)–(3.18) one makes an analytic continuation in the variable s to complex values. If one chooses for b quantum numbers that also change when s is analytically continued, b could not be kept at one and the same value during this analytic continuation and the Gamow vector on the lhs of (3.20) would be a complicated (continuous) superposition (integral) over different values of b and not just a superposition over different values of s. For this reason, the momentum $\mathbf{p}$ is not a good choice for the quantum numbers b, because the momentum will also become

[9] That it is important not to use asymptotically-free states for resonances has also been emphasized by Sirlin [24].

complex if the energy in the centre of mass rest frame becomes complex. This is also the reason why we choose the space components of the four-velocity $\hat{\mathbf{p}} = \mathbf{p}/\sqrt{\mathsf{s}}$ as the additional quantum numbers in definition (2.9). The momenta $\mathbf{p}$ will become complex in the analytic continuation in such a way that $\hat{p}^\mu = p^\mu/\sqrt{\mathsf{s}}$ will always be real. This condition restricts the arbitrariness of the analytic continuation, it makes the momentum only 'minimally complex' and keeps the representations of the Lorentz subgroup of the Poincaré group $\mathcal{P}$ unitary. The homogeneous Lorentz transformations $U(\Lambda)$ are the same as in Wigner's representations. We will call this subclass of semigroup representations of $\mathcal{P}$ minimally complex [25, 26].

With (3.19) and (3.20) we have obtained for each resonance defined by the pole of the jth partial S-matrix at $\mathsf{s} = \mathsf{s}_R$ an 'exact' Breit–Wigner (3.19) and associated with it a set of 'exact' Gamow kets (3.20). These Gamow kets (3.20) span the space of an irreducible representation $[j, \mathsf{s}_R]$ of Poincaré transformations. Thus we have the obtained correspondence (2.13) between the 'exact Breit–Wigner' and 'exact Gamow vectors' from their common origin— the first-order pole of the S-matrix at s_{R_i}.

For the association (2.13) of a representation space $[j, \mathsf{s}_R]$ with a phenomenological partial wave amplitude $a_j^{\mathrm{res}}(\mathsf{s})$, we had used the equality of (1.4) and (3.19) which has a very special property, namely it is a Cauchy kernel. The definition of the vectors (3.20) would not have been possible for other arbitrary functions of s (e.g. not for the amplitude $a_j^{om}(\mathsf{s})$ of (1.3)). Even for the Breit–Wigner (1.4) we had to extend the values of s from the phenomenologically testable values $m_0^2 \leqslant \mathsf{s} < \infty$ to the negative axis and introduce an idealization, the 'exact' Breit–Wigner (3.19) for which s extends over $-\infty_{II} < \mathsf{s} < +\infty$. Only for the exact Breit–Wigner (3.19) could we use the Titchmarsh theorem in (3.18) and associate with the amplitude $a_j^{\mathrm{BW}}(\mathsf{s})$ a vector which is defined by this exact Breit–Wigner amplitude. In order to apply the Titchmarsh theorem we had to restrict the admissible wavefunctions $\overline{\psi^-}(\mathsf{s})$ and $\phi^+(\mathsf{s})$ to be Hardy class in the lower half plane. This means we had to require the hypothesis (3.9), (3.10), or in other words had to require that the in-state vector ϕ^+ and the out-observable vector ψ^- that can appear in the S-matrix element (3.7) and (3.1) be in the spaces $\mathbf{\Phi}_-$ and $\mathbf{\Phi}_+$, respectively. Only then could we define the Gamow kets $|[j, \mathsf{s}_R], b^-\rangle$ in terms of the Dirac–Lippmann–Schwinger kets $|[j, \mathsf{s}], b^-\rangle$ by (2.9) as generalized vectors or functionals over the Hardy class space $\mathbf{\Phi}_+$. The Gamow vectors cannot be defined as functionals over the Schwartz space $\mathbf{\Phi}$ as the usual Dirac kets[10]. Thus the new hypothesis of (2.5), (2.6) had to be invoked in order to obtain for every S-matrix pole a Breit–Wigner amplitude (3.19) and a corresponding Gamow vector (3.20). Each Gamow vector (3.20) by itself has a precise exponential time evolution as shown by (2.26) and if one could prepare a pure Gamow state, i.e. isolate the $\psi^G_{\mathsf{s}_{R_i}}$ associated with the resonance pole at s_{R_i}, then the probability for the decay products would be given precisely by the exponential decay law with the lifetime $\tau_{R_i} = \frac{\hbar}{\Gamma_{R_i}}$.

4. Superposition of Gamow vectors for the jth partial wave and deviations from the exponential decay law

Each exact Breit–Wigner amplitude $a_j^{\mathrm{BW}_i}(\mathsf{s})$ of (3.19) was one particular part of the jth partial S-matrix (3.14), namely the one that was obtained from the pole of $S_j(\mathsf{s})$ at s_{R_i} and considered as a separate entity in section 3. With our mathematical hypothesis of (2.5), (2.6), it was completely natural to treat each integral around a resonance pole at s_{R_i} separately and assign to each a Breit–Wigner amplitude (3.19) and a corresponding Gamow vector (3.20). In an

[10] Similarly, we can define another kind of Gamow ket $|[j, \mathsf{s}_R^*], b^+\rangle \in \mathbf{\Phi}_-^\times$ in terms of the Dirac–Lippmann–Schwinger kets $|[j, \mathsf{s}], b^+\rangle$ for the resonance pole at $\mathsf{s}_R^* = (M_R + \mathrm{i}\Gamma/2)^2$ in the upper half plane of the second sheet.

experiment it is not that easy to separate the resonance term(s) from the remainder of the S-matrix, since the jth partial cross section contains not only the resonance part but also the non-resonant background of the scattering process in the amplitude $a_j(\mathsf{s})$. We now want to consider the whole S-matrix element (3.11). Inserting (3.17) and the definition (3.12) into (3.11) we can write the jth partial S-matrix element (ψ^-, ϕ^+) as a discrete sum over Gamow vectors and the background term,

$$(\psi^-, \phi^+) = \langle \psi^- | \phi^{bg} \rangle + \sum_i \left\langle \psi^- \middle| \mathsf{s}_{R_i}^- \right\rangle (2\pi R^{(i)}/\mathrm{i}) \left\langle {}^+\mathsf{s}_{R_i} \middle| \phi^+ \right\rangle. \tag{4.1}$$

Omitting the arbitrary $\psi^- \in \Phi_+$ (the observable) one writes equation (4.1) as a functional equation in the space $\Phi_+^\times$ and obtains the following expansion of the prepared in-state $\phi^+ \in \Phi_-$:

$$\phi^+ = \phi^{bg} + \sum_i \left| \mathsf{s}_{R_i}^- \right\rangle c_{R_i} \qquad \text{where} \quad c_{R_i} = (2\pi R^{(i)}/\mathrm{i}) \left\langle {}^+\mathsf{s}_{R_i} \middle| \phi^+ \right\rangle. \tag{4.2}$$

In this way the in-state ϕ^+ has been decomposed into a vector representing the non-resonant part ϕ^{bg} and a sum over the Gamow vectors each representing a resonance state. The complex eigenvalue expansion (4.2) is an alternative generalized eigenvector expansion to Dirac's eigenvector expansion (2.8), which in this abbreviated notation reads

$$\Phi_- \ni \phi^+ = \int_0^\infty \mathrm{d}\mathsf{s} |\mathsf{s}^+\rangle \langle {}^+\mathsf{s} | \phi^+ \rangle \qquad |\mathsf{s}^+\rangle \in \Phi_-^\times. \tag{4.3}$$

While equation (4.3) expresses the in-state ϕ^+ in terms of the Lippmann–Schwinger kets $|\mathsf{s}^+\rangle \in \Phi_-^\times$, which are generalized eigenvectors of the mass operator $P_\mu P^\mu$ with real eigenvalue s, equation (4.2) is an expansion of $\phi^+ \in \Phi_+^\times$ in terms of eigenkets $\left|\mathsf{s}_{R_i}^-\right\rangle \in \Phi_+^\times$ of the same *self-adjoint* mass operator $P_\mu P^\mu$ with complex generalized eigenvalue $\mathsf{s}_{R_i} = \left(M_{R_i} - \mathrm{i}\Gamma_{R_i}/2\right)^2$ and the vector ϕ^{bg} of (3.13).

The term ϕ^{bg} is defined by (3.12), (3.13) and is therefore an element of $\Phi_+^\times$. We want to rewrite (3.12) in a more familiar form. According to the van Winter theorem [27], a Hardy class function on the negative real axis is uniquely determined by its values on the real positive axis (cf appendix A2 of [28]). Therefore one can use the Mellin transform to rewrite the integral on the lhs of (3.12) as an integral over the interval $m_0^2 \leqslant \mathsf{s} < \infty$ and obtain

$$\begin{aligned} \langle \psi^- | \phi^{bg} \rangle &= \int_{m_0^2}^{-\infty_{II}} \mathrm{d}\mathsf{s} \langle \psi^- | \mathsf{s}^- \rangle S_j(\mathsf{s}) \langle {}^+\mathsf{s} | \phi^+ \rangle \\ &= \int_{m_0^2}^{\infty} \mathrm{d}\mathsf{s} \langle \psi^- | \mathsf{s}^- \rangle b_j(\mathsf{s}) \langle {}^+\mathsf{s} | \phi^+ \rangle \end{aligned} \tag{4.4}$$

where $b_j(\mathsf{s})$ is uniquely defined by the values of $S_j(\mathsf{s})$ on the negative real axis. Without more specific information about $S_j(\mathsf{s})$, we cannot be certain about the energy dependence of the background $b_j(\mathsf{s})$. If there are no further poles or singularities besides those included in the sum, then $b_j(\mathsf{s})$ is likely to be a slowly varying function of s [29]. Omitting the arbitrary $\psi^- \in \Phi_+$ we write the expansion for the non-resonant background part ϕ^{bg} of the prepared in-state vector ϕ^+ as

$$|\phi^{bg}\rangle = \int_{m_0^2}^{\infty} \mathrm{d}\mathsf{s} |\mathsf{s}^-\rangle \langle {}^+\mathsf{s} | \phi^+ \rangle b_j(\mathsf{s}). \tag{4.5}$$

Inserting (4.5) into (4.2) we obtain the complex basis vector expansion of every $\phi^+ \in \Phi_-$ as

$$\phi^+ = \sum_i \left| \mathsf{s}_{R_i}^- \right\rangle c_{R_i} + \int_{m_0^2}^{\infty} \mathrm{d}\mathsf{s} |\mathsf{s}^-\rangle \langle {}^+\mathsf{s} | \phi^+ \rangle b_j(\mathsf{s}) \qquad \left| \mathsf{s}_{R_i}^- \right\rangle, |\mathsf{s}^-\rangle \in \Phi_+^\times \tag{4.6}$$

which is a functional equation over the space Φ_+, cf (4.1).

The basis vector expansion (4.6) shows that the resonances appear here on the same footing as the bound states in the usual basis vector expansion for a system with discrete energy eigenvalues[11], with the only difference that the bound states are represented by proper vectors $|E_n) \in \mathcal{H}$ and the Gamow states are represented by generalized vectors, $|\mathsf{s}_{R_i}^-\rangle \in \Phi_+^\times$. The basis vector expansion (4.6) shows that in addition to the superposition of N Gamow states there appears an integral (or continuous superposition) over the continuous basis vectors $|s^-\rangle$ with a weight function $b(s)\langle^+s|\phi^+\rangle$, where the wavefunction $\phi^+(s) = \langle^+s|\phi^+\rangle$ depends upon the particular preparation of the state ϕ^+ and will change with the preparation from experiment to experiment.

In order to study the evolution of ϕ^+ we apply a Poincaré transformation $U^\times(\Lambda, x)$ as in (2.24) to the functional equation (4.6). For the sake of simplicity we consider the special case of the time evolution in the rest frame, i.e. we choose

$$\begin{gathered}|\mathsf{s}_{R_i}^-\rangle = |[j, \mathsf{s}_{R_i}]\, j_3 \hat{\mathbf{p}} = 0^-\rangle \qquad \text{and} \qquad |\mathsf{s}^-\rangle = |[j, \mathsf{s}]\, j_3\hat{\mathbf{p}} = 0^-\rangle \\ \text{with} \quad H^\times|\mathsf{s}^-\rangle = \sqrt{\mathsf{s}}|\mathsf{s}^-\rangle\end{gathered} \tag{4.7}$$

and

$$U^\times(\Lambda, x) = U^\times(\Lambda = 1, x = (t, 0, 0, 0)) = \mathrm{e}^{-\mathrm{i}H^\times t}. \tag{4.8}$$

Then we obtain for the time evolution of the state $\phi^+ \in \Phi_- \subset \Phi_+^\times$

$$\phi^+(t) = \mathrm{e}^{-\mathrm{i}H^\times t}\phi^+ = \sum_{i=1}^{N} \mathrm{e}^{-\mathrm{i}M_{R_i}t}\,\mathrm{e}^{-\Gamma_{R_i}/2t}|\mathsf{s}_{R_i}\rangle + \int_{m_0^2}^{\infty} \mathrm{e}^{-\mathrm{i}\sqrt{\mathsf{s}}t}|\mathsf{s}^-\rangle\langle^+\mathsf{s}|\phi^+\rangle b(\mathsf{s}) \qquad \text{for} \quad t \geqslant 0 \tag{4.9}$$

where we have used (4.7) and (2.26). The result (4.9) shows that the time evolution of a state ϕ^+ prepared by an apparatus is given by a superposition of exponentials plus a non-exponential background integral.

The complex basis vector expansion (4.6) is an exact consequence of the new hypothesis (2.6), (2.7). In the heuristic treatment of the decay phenomena by a complex effective Hamiltonian based on the Weisskopf–Wigner approximation [30] like the Lee–Oehme–Yang theory of the neutral Kaon system [32], or the effective theories with finite-dimensional complex Hamiltonian matrices in nuclear physics [33], one always ignores the continuum terms. These theories correspond to (4.6) with the background integral omitted:

$$\phi^+ \approx \sum |\mathsf{s}_{R_i}^-\rangle c_i = |K_S\rangle c_S + |K_L\rangle c_L. \tag{4.10}$$

On the far right of (4.10) we used the standard notation for the K^0-state expressed in terms of the K_L- and K_S-states which are the eigenstates of the complex Hamiltonian matrix with complex energy $(M_L - \mathrm{i}\Gamma_L/2)$ and $(M_S - \mathrm{i}\Gamma_S/2)$, respectively. In this case the probability for the decay products $\pi^+\pi^-$ described by $\psi^- = |\pi^+\pi^-\rangle$ is obtained from (4.9) as

$$\begin{aligned}|\langle\psi^-|\phi^+(t)\rangle|^2 &= |c_S|^2\,\mathrm{e}^{-\Gamma_S t}|\langle\psi^-|K_S\rangle|^2 + |c_L|^2\,\mathrm{e}^{-\Gamma_L t}|\langle\psi^-|K_L\rangle|^2 \\ &\quad + 2|c_L \cdot c_S|\,\mathrm{e}^{-\frac{1}{2}(\Gamma_S+\Gamma_L)t}|\langle\psi^-|K_S\rangle\langle\psi^-|K_L\rangle|\cos((M_S - M_L)t + \alpha).\end{aligned} \tag{4.11}$$

In comparison to the decay probability rate (2.27) for the Gamow vectors, the decay rate for the superposition (4.10) shows deviations from the exponential decay law due to the $K_S - K_L$ interference terms of (4.11). These deviations from the exponential due to the interference of two exponentially decaying states are well known and well accepted. However,

[11] We have assumed as in (4.3) that there are no bound states of H. Otherwise one would have in addition to the rhs of (4.3) and (4.6) the discrete sum over the bound states, which is orthogonal to the rest $\phi = \sum_n |E_n)(E_n|\phi) + \int_0^\infty \mathrm{d}E|E\rangle\langle E|\phi\rangle$.

even if there is only one resonance in the jth partial wave then the exact complex basis vector expansion (4.6) is given by

$$\phi^+ = \psi^G + \int \mathrm{ds}|\mathsf{s}^-\rangle\langle^+\mathsf{s}|\phi^+\rangle b(\mathsf{s}) = \psi^G + \phi^{BG}. \tag{4.12}$$

Here ψ^G is the Gamow vector with the exponential time evolution (2.26) and the prepared in-state ϕ^+ thus has the time evolution

$$\phi^+(t) = \mathrm{e}^{-\mathrm{i}H^\times t}\phi^+ = \mathrm{e}^{-\mathrm{i}m_R t}\,\mathrm{e}^{-\Gamma_R/2t}\psi^G + \int_{m_0^2}^{\infty} \mathrm{e}^{-\mathrm{i}\sqrt{\mathsf{s}}t}|\mathsf{s}^-\rangle\langle^+\mathsf{s}|\phi^+\rangle b(\mathsf{s}) \tag{4.13}$$

which predicts deviations from the exponential law due to the complicated time dependence of the background vector at time t:

$$\phi^{bg}(t) = \int_{m_0^2}^{\infty} \mathrm{e}^{-\mathrm{i}\sqrt{\mathsf{s}}t}|\mathsf{s}^-\rangle\langle^+\mathsf{s}|\phi^+\rangle b(\mathsf{s}). \tag{4.14}$$

Only the Gamow vector, which corresponds by (2.13) to the exact Breit–Wigner amplitude and represents the resonance *per se*, has a purely exponential decay (2.26). The time evolution of the first term in (4.13) is in every experiment given by the same exponential, which is for the particular decaying Gamow state characterized by two numbers (M_R, Γ_R) only. The time evolution of the second term of (4.13)—and also of the interference term of $\psi^G(t)$ and $\phi^{bg}(t)$—depends upon the preparation of the state ϕ^+ and changes thus from experiment to experiment. The smaller one can make the experimentally controllable quantity $|\langle^+\mathsf{s}|\phi^+\rangle|$, the less important will be the deviations from the experimental law. In some experiments, using a suitable analysis that separates the interference term, the validity of the exponential law has been established to a high degree of accuracy [34]. If one takes the hypothesis that the experimentally prepared state must be $\phi^+ \in \Phi_-$ (and cannot be $\psi^G \in \Phi_+^\times$), then there must always be a ϕ^{bg} in ϕ^+ of (4.12) even though ϕ^+ may come arbitrarily close (with respect to the definition of convergence in the space $\Phi_+^\times$, $\Phi_- \subset \Phi_+^\times$) to an exponentially decaying generalized state ψ^G [35][12]. In resonance scattering experiments of hadrons, in which the timescale for the preparation of the decaying state and the timescale for the decay are the same ($\approx 10^{-23}$ s), one knows from experiments that one always needs the slowly varying background term $B_j(\mathsf{s})$ in the scattering amplitude (1.2) for the fit of the cross-section data. Therefore one expects deviations from the exponential decay law due to $\langle^+\mathsf{s}|\phi^+\rangle b(\mathsf{s})$ in (4.14), since $b(\mathsf{s})$ corresponds to the background amplitude $B_j(\mathsf{s})$, as we shall show below. This background term can also account for the first (and the only one reported so far) experimental observation of deviations from the exponential law [36]. It is thus our complex basis vector expansion (4.6) based on the new hypothesis (2.5), (2.6) that explains the deviations from the exponential law and relates it to the background in a resonance scattering experiment. Though the resonance *per se* described by a Gamow vector has an exponential time evolution, the complex basis vector expansion predicts deviations from the exponential law for the prepared state if the background cannot be separated from the resonance *per se*.

[12] The historical origin of the much discussed deviations from the time-honoured exponential decay law is a theorem of Hilbert space mathematics using the specific topological properties of the Hilbert space, namely completion. One proves the theorem that the survival probability and thus the partial decay rate $\dot{P}_\eta(t)$ (which is measured as the counting rate $\dot{N}_\eta(t)$) for every state described by $h \in \mathcal{H}$ cannot be exponential (cf [35]). Since our in-state $\phi^+ \in \Phi_- \subset \mathcal{H}$ is also an element of $\mathcal{H}$ and thus cannot have an exponential time evolution this is expressed by the background in (4.13). In practical calculations one uses many concepts, such as the Lippmann–Schwinger kets, that lie outside the Hilbert space, and so does the Gamow vector. The theorem from Hilbert space mathematics is no reason why the vector which describes the resonance *per se* needs to be an element of $\mathcal{H}$ and, therefore, need not have deviations from the exponential decay law.

The experimental ingenuity in establishing the exponential decay law for each Gamow state is to suppress or exclude by analysis as much as possible the effect of this background and the effect of the other interfering Gamow vectors.

We now want to establish the correspondence between the terms in the complex basis vector expansion (4.6) and terms in the scattering amplitude. For this purpose we rewrite (3.11) in a different form. In place of (ψ^-, ϕ^+) on the lhs of (3.11), we write the rhs of (3.8), and on the rhs of (3.11) we use (3.18) and (4.4). Then we obtain

$$\int_{m_0^2}^{\infty} \mathsf{ds}\langle\psi^-|\mathsf{s}^-\rangle S_j(\mathsf{s})\langle^+\mathsf{s}|\phi^+\rangle = \int_{m_0^2}^{\infty} \mathsf{ds}\langle\psi^-|\mathsf{s}^-\rangle b_j(\mathsf{s})\langle^+\mathsf{s}|\phi^+\rangle + \sum_i \int_{-\infty_{II}}^{+\infty} \mathsf{ds}\langle\psi^-|\mathsf{s}^-\rangle\langle^+\mathsf{s}|\phi^+\rangle \frac{R^{(i)}}{\mathsf{s}-\mathsf{s}_{R_i}}. \tag{4.15}$$

This equality holds for the whole space of functions $\langle\psi^-|\mathsf{s}^-\rangle\langle^+\mathsf{s}|\phi^+\rangle \in \Phi_-$. Therefore we can omit these arbitrary energy wavefunctions

$$\langle\psi^-|\mathsf{s}^-\rangle\langle^+\mathsf{s}|\phi^+\rangle = \langle\psi^{\mathrm{out}}|\mathsf{s}\rangle\langle\mathsf{s}|\phi^{\mathrm{in}}\rangle \in \Phi_- \tag{4.16}$$

which describe the resolution of the preparation apparatus and of the registration apparatus and write equation (4.15) as an equation between distributions over the function space Φ_-:

$$\theta\left(\mathsf{s}-m_0^2\right) S_j(\mathsf{s}) = \theta\left(\mathsf{s}-m_0^2\right) b_j(\mathsf{s}) + \sum_i \frac{R^{(i)}}{\mathsf{s}-\mathsf{s}_{R_i}}. \tag{4.17}$$

Though one likes to represent the 'physics' in this apparatus-independent way, what one measures in each experiment contains of course always the convolution with an apparatus resolution so that (4.17) really means (4.15) in its applications to a particular experiment. A corresponding equation can be written for the partial wave amplitudes (3.6) (by dividing (4.17) by 2i and subtracting 1 on both sides for $n = n'$)

$$\theta\left(\mathsf{s}-m_0^2\right) a_j(\mathsf{s}) = \theta\left(\mathsf{s}-m_0^2\right) B_j(\mathsf{s}) + \sum_i \frac{\tilde{R}^{(i)}}{\mathsf{s}-\mathsf{s}_{R_i}} \tag{4.18}$$

considered as a functional equation in the space of distributions $\Phi_-^\times$. Here $B_j(\mathsf{s})$ as $b_j(\mathsf{s})$ describes an ever present slowly varying non-resonant background. For instance, if there is only one resonance pole at s_R in the jth partial wave, then the cross section contains in addition to the resonance part (Breit–Wigner) also the background amplitude and the interference term between them:

$$|a_j(\mathsf{s})|^2 = \left|B_j(\mathsf{s}) + \frac{\tilde{R}^{(i)}}{\mathsf{s}-\mathsf{s}_{R_i}}\right|^2 = \left|B_j(\mathsf{s}) + a_j^{\mathrm{res}}(\mathsf{s})\right|^2. \tag{4.19}$$

This shows that it is difficult to distinguish phenomenologically between two alternative functions for the resonance part of the amplitude such as (1.3) and (1.4). The formula (4.19), or the formula (4.18) for a single resonance, is one of the frequently used phenomenological formulae. This formula has been derived here from the hypothesis (2.5), (2.6) but it also has a theoretical justification under the usual analyticity assumption for the S-matrix $S_j(\mathsf{s})$ (Laurent expansion).

For two (or more) resonances we have also derived (4.17), but we needed for its derivation the Hardy class hypothesis (3.9), (3.10) to obtain (3.18) for each pole term separately. The amplitude (4.17), (4.18) represents the superposition of interfering Breit–Wigner resonances plus a background. The formula (4.17) cannot be derived using the usual analyticity assumption of $S_j(\mathsf{s})$ only [9]. The interference of resonances as predicted by (4.17), (4.18) has been established experimentally for the non-relativistic case in nuclear physics [31][13] and for the relativistic case in the ρ–ω interference [10]. Since its derivation required the assumption $\psi^- \in \mathbf{\Phi}_+$ and $\phi^+ \in \mathbf{\Phi}_-$, the phenomenological success of formulae such as (4.17) for two interfering resonances is another argument in favour of our new hypothesis of (2.5), (2.6).

With (4.17) and (4.6) we have now established a term-by-term correspondence between the complex eigenvalue resolution (4.6) of the prepared in-state vector ϕ^+ and the representation (4.18) for the partial wave amplitude (or also (4.17) for the S-matrix). To each Breit–Wigner in (4.18) corresponds a Gamow vector in (4.2) or (4.6) and to the background amplitude $B_j(\mathsf{s})$ in (4.18) corresponds the vector ϕ^{bg}. This establishes a unique correspondence between the vector description of quasistable particles and the S-matrix description of resonances. The vector description is used for instance in the effective theories with complex Hamiltonian matrix (such as the Lee–Oehme–Yang theory of K_S^0 and K_L^0 [32] or the finite-dimensional models of nuclear physics [31, 33]), only these finite-dimensional models omit the background vectors ϕ^{bg} (4.5) which span an infinite-dimensional energy continuum. This omission of the energy continuum is a typical feature of the Weisskopf–Wigner approximations. In the S-matrix description of the resonance by the phenomenological ansatz (1.2) one usually does not omit the background amplitude $B(\mathsf{s})$, but often includes in $B(\mathsf{s})$ also the contribution of a second distant resonance which according to our prediction (4.18) belongs to the sum of the Breit–Wigner amplitude. The correspondence between (4.18) and (4.6) unifies the theory of resonance scattering and the theory of particle decay.

5. Summary and conclusion

In this paper a relativistic theory that unifies resonance phenomena and decay phenomena has been presented. The centrepiece of this theory is the relativistic Gamow vector which is defined as the vector with an 'ideal' Breit–Wigner energy wavefunction. Each Gamow vector is obtained from a resonance pole, in the second sheet of the analytically continued relativistic S-matrix at $\mathsf{s}_R = (M_R - \mathrm{i}\Gamma_R/2)^2$ (figure 1), where M_R is the resonance mass and Γ_R is the width of the resonance part (1.4) of the scattering amplitude.

A vector of this kind does not exist in the Hilbert space. To accommodate it, the Hilbert space axiom of quantum theory had to be replaced by a new hypothesis which distinguishes meticulously between prepared in-states and detected out-observables by associating them with different Hardy spaces (2.5) and (2.6) which are dense in the same Hilbert space. The Gamow states are represented by elements in the space of continuous antilinear functionals on the Hardy space. A consequence of this new hypothesis is that the quantum theory of scattering and decay is time asymmetric (expressing irreversibility on the microphysical level), and the time evolution of the Gamow vectors is given by a semigroup, $0 < t < \infty$, where $t = 0$ is the time at which the Gamow state was created, and at which the registration of the decay products can begin. For each decay event at t_i, there is a time t_{i_0} at which the decaying state was created and we associate each t_{i_0} with the mathematical semigroup time $t = 0$.

The relativistic Gamow kets (2.9) are the basis vectors of a semigroup representation of the causal Poincaré transformations (2.15) characterized by $[j, \mathsf{s}_R]$ representing spin j of the resonating partial wave and complex pole position s_R. In order to retain as much similarity as possible with Wigner's unitary representations of the Poincaré group for stable particles $[j, m^2]$ and to maintain the meaning of spin j of a resonance, only semigroup representations with minimally complex momentum are considered, which means that the momentum is given by $\mathbf{p} = \sqrt{\mathsf{s}_R}\hat{\mathbf{p}}$, where the four-velocities $\hat{p}^0 = \gamma = \sqrt{1 - \mathbf{v}^2}$, $\hat{\mathbf{p}} = \gamma\mathbf{v}$ are real. The general

[13] This paper and also [32] use the finite complex effective Hamiltonian omitting the background ϕ^{bg} and/or $b_j(\mathsf{s})$.

transformation formula of the relativistic Gamow kets under causal Poincaré transformations is given by (2.24) which looks very similar to Wigner's unitary Poincaré group transformation, but differs by the property that transformations are allowed only into the forward light cone. (A surprising side result is that also the in- and out-plane wave solutions of the Lippmann–Schwinger equation, when given a proper mathematical meaning as functionals on linear topological spaces, support only a semigroup representation of Poincaré transformations and not a unitary representation of the Poincaré group as has been universally assumed, cf e.g. [13]. The reason for this is the infinitesimal imaginary part $\mp i\epsilon$ in the energy of the Lippmann–Schwinger kets expressing time asymmetric boundary conditions.) For the case of an isolated Gamow resonance state at rest, (2.24) gives the exponential time evolution (2.26) which leads to the exponential decay probability (2.27) with lifetime $\tau_R = \hbar/\Gamma_R$.

The prepared in-state ϕ^+ is in general not given by a Gamow vector but is a linear superposition of Gamow vectors for all the N resonance poles of the jth partial S-matrix $S_j(s)$ and in addition there is a background vector so that the prepared state is given by the complex basis vector expansion (4.2) or (4.6), where the sum over i extends over all N resonance poles of the jth partial wave. This is very similar to the heuristic expansion in terms of eigenstates with complex eigenvalues $\left(M_i - \frac{\Gamma_{R_i}}{2}\right)$ of a finite-dimensional effective Hamiltonian, such as the K_0-state in (4.10). However, our exact complex basis vector expansion has in addition to the superposition of states with definite lifetime τ_i, also a background vector (4.5) over the energy continuum, which is always lost in the Weisskopf–Wigner approximation. As the consequence of this background term, which corresponds to the slowly varying background amplitude of the scattering amplitude (1.2), one always obtains deviations from the exponential decay law for the prepared state ϕ^+. This explains that in spite of the exponential time evolution for the Gamow state, describing the resonance *per se*, one can observe deviations from the exponential law even if there is only one resonance present. Whereas the resonance *per se*, which does not depend upon the experiment that prepares the state, is characterized by (M_R, Γ_R) and has exponential evolution with lifetime $\tau_R = 1/\Gamma_R$, the prepared state can change from experiment to experiment and so would the deviation from the exponential decay law. Eliminating the background in the analysis of the decay data for each particular experiment should reveal the exponential character of the decay.

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Appendix. Rigged Hilbert spaces

Rigged Hilbert spaces (RHS), also called Gelfand triplets, are triplets of linear spaces, which differ from each other by their topology. In other words the meaning of convergence is different for each space, which implies that the limit points of converging sequences are different in the three spaces that make up the Gelfand triplet. The three spaces have properties which a physicist would call a Hilbert space but only one of them is a Hilbert space by the mathematical definition, i.e. it is complete with respect to the Hilbert space convergence.

One starts with a linear scalar product space denoted by Φ_{alg} (also called a pre-Hilbert space). The subscript (alg) refers to the algebraic operations that one can perform in it, namely linear superpositions and the scalar product. The three spaces that form the RHS, denoted by

$$\Phi \subset \mathcal{H} \subset \Phi^{\times} \tag{A.1}$$

are obtained by completing the purely algebraic space Φ_{alg} with respect to three different topologies, i.e. three definitions of convergence. To obtain each space, one adjoins to Φ_{alg} the limit elements of Cauchy sequences, but one uses three different meanings of convergence and thus obtains three different complete spaces. The space with a stronger topology, i.e. a stronger definition of convergence, is dense in the space with a weaker topology. The Hilbert space $\mathcal{H}$ is obtained by completing Φ_{alg} with respect to the norm, denoted by $\tau_{\mathcal{H}}$. The space Φ is obtained by completing Φ_{alg} with respect to a stronger topology than $\tau_{\mathcal{H}}$, denoted by τ_{Φ}. The third space $\Phi^{\times}$ is the space of continuous antilinear functionals F on Φ

$$|F\rangle : \phi \in \Phi \to F(\phi) = \langle\phi|F\rangle \in \mathbb{C}. \tag{A.2}$$

Thus one obtains the triplet of spaces, or a rigged Hilbert space (A.1).

In the Hilbert space, there is a one-to-one correspondence between elements of the space of antilinear functionals $\mathcal{H}^{\times}$ and elements of $\mathcal{H}$, thus one can identify them with each other:

$$\mathcal{H} = \mathcal{H}^{\times}. \tag{A.3}$$

According to the Frechet–Riesz theorem, for every $|f\rangle \in \mathcal{H}^{\times}$ there is an $f \in \mathcal{H}$ such that $f(\phi) = \langle\phi|f\rangle = (\phi, f)$ for all $\phi \in \mathcal{H}$. The functional $\langle\phi|F\rangle$ is an extension of the scalar product (ϕ, f) to those $|F\rangle \in \Phi^{\times}$ which are not in $\mathcal{H}$.

Let A be a linear operator in Φ, continuous with respect to τ_{Φ}, and $A^{\dagger}$ its adjoint in $\mathcal{H}$. To the triplet of spaces (A.1) corresponds to a triplet of operators

$$A : A^{\dagger}|_{\Phi} \subset A^{\dagger} \subset A^{\times} \tag{A.4}$$

where $A^{\dagger}$ is the Hilbert space adjoint of A and $A^{\dagger}|_{\Phi}$ its restriction to Φ. If A is a continuous operator with respect to τ_{Φ}, it need not be, and in general is not, a continuous (bounded) operator in $\mathcal{H}$. We shall only consider τ_{Φ}-continuous operators. So far the restriction to continuous operators in Φ has proved to be sufficient for quantum physics, whereas $\tau_{\mathcal{H}}$-continuous operators are not sufficient (e.g. the position and/or momentum operators cannot be continuous operators in $\mathcal{H}$, neither can the generators of unitary representations of non-compact groups).

The conjugate operator, $A^{\times}$, of the τ_{Φ}-*continuous* linear operator A is a continuous linear operator in $\Phi^{\times}$ defined by

$$\langle A\phi|F\rangle = \langle\phi|A^{\times}|F\rangle \qquad \forall\phi \in \Phi \quad \text{and} \quad \forall F \in \Phi^{\times}. \tag{A.5}$$

It is a unique extension of the Hilbert space adjoint operator

$$(A\phi, f) = (\phi, A^{\dagger} f) \qquad \text{for} \quad \phi, f \in \mathcal{H}. \tag{A.6}$$

A vector $F \in \Phi^{\times}$ is called a generalized eigenvector of the τ_{Φ}-continuous operator A if for some $\omega \in \mathbb{C}$

$$\langle A\phi|F\rangle = \langle\phi|A^{\times}|F\rangle = \omega\langle\phi|F\rangle. \tag{A.7}$$

This is also written as $A^{\times}|F\rangle = \omega|F\rangle$, or as Dirac did, $A|F\rangle = \omega|F\rangle$ for Hermitian A. An example of generalized eigenvectors is the Dirac kets. Their eigenvalues belong to the continuous spectrum of a self-adjoint H, $H^{\times}|E\rangle = E|E\rangle$, $0 \leqslant E < \infty$.

Kets (and all $F \in \Phi^{\times}$) depend on the choice of the space Φ. Dirac kets are usually defined with Φ as the Schwartz space S, i.e. the space of smooth, rapidly decreasing wavefunctions $\phi(E) = \langle E|\phi\rangle$.

The triplet of function spaces

$$S \subset L^2 \subset S^\times \tag{A.8}$$

where S is the space of smooth, rapidly decreasing functions and L^2 is the space of Lebesgue square-integrable functions with scalar product given by the integral

$$(\psi, \phi) = \int_{-\infty}^{\infty} \mathrm{d}E\,\overline{\psi(E)}\phi(E) = \int_{-\infty}^{\infty} \mathrm{d}E\,\langle\psi|E\rangle\langle E|\phi\rangle \tag{A.9}$$

is an example of a RHS. It is called a realization of the abstract Schwartz-RHS

$$\Phi \subset \mathcal{H} \subset \Phi^\times \tag{A.10}$$

whose vectors $\psi, \phi \in \Phi$ are the vectors for which the Dirac basis vector expansion

$$\phi = \int_{-\infty}^{\infty} \mathrm{d}E\,|E\rangle\langle E|\phi\rangle \tag{A.11}$$

(omitting the arbitrary $\psi \in \Phi$ from (A.9)) holds.

The integrals in (A.9) can always be chosen as Riemann integrals if $\psi, \phi \in \Phi$, i.e. $\psi(E), \phi(E) \in S$ and we shall do so. The integral for the scalar product in L^2, however, must be a Lebesgue integral since the space of Riemann square-integrable functions cannot be a complete Hilbert space.

The space Φ that together with $\mathcal{H}$ forms an RHS cannot be an arbitrary topological space. The topology τ_Φ must fulfil certain additional conditions (e.g. nuclearity) in order that Dirac's basis vector expansion (A.11) can be proved as the nuclear spectral theorem. These additional requirements on Φ are part of the definition of every RHS (A.1). The Dirac basis vector expansion (A.11) is the most important theorem for quantum mechanics; even before its proof, it had been used profusely in quantum theory. In this paper it appears in (2.2)–(2.7).

Examples of other RHS besides the Schwartz-RHS (A.8) are the Hardy-RHS. There are two Hardy-RHS, denoted by

$$\Phi_+ \subset \mathcal{H} \subset \Phi_+^\times \tag{A.12}$$

$$\Phi_- \subset \mathcal{H} \subset \Phi_-^\times \tag{A.13}$$

and realized by the function spaces

$$\mathsf{H}^2_+ \cap S|_{\mathbb{R}_+} \subset L^2(\mathbb{R}_+) \subset \left(\mathsf{H}^2_+ \cap S|_{\mathbb{R}_+}\right)^\times \tag{A.14}$$

$$\mathsf{H}^2_- \cap S|_{\mathbb{R}_+} \subset L^2(\mathbb{R}_+) \subset \left(\mathsf{H}^2_- \cap S|_{\mathbb{R}_+}\right)^\times \tag{A.15}$$

respectively.

Here the Hilbert space, $L^2(\mathbb{R}_+)$, is the space of Lebesgue square-integrable functions on the positive real line $\mathbb{R}_+$, and $\mathsf{H}^2_\pm \cap S|_{\mathbb{R}_+}$ denotes the smooth, rapidly decreasing functions $\psi^\pm(E)$, $E \in \mathbb{R}_+$, which can be analytically continued into the upper half (for H^2_+) and the lower half (for H^2_-) complex energy planes. More precisely, the $\psi^\mp(E) \in \mathsf{H}^2_\pm \cap S|_{\mathbb{R}_+}$ are the boundary values of smooth analytic functions in the lower (−) and upper (+) complex half planes that decrease sufficiently fast at the infinite semicircle (for the definition see the appendix of [38]). We call the spaces $\Phi_\pm$ and their realization $\mathsf{H}^2_\pm \cap S|_{\mathbb{R}_+}$ Hardy spaces. One can show that these function spaces (A.14), (A.15), also form an RHS [19]. The Hardy-RHS are needed if one wants to consider generalized eigenvectors of the Hamiltonian H belonging to the continuous spectrum

$$H^\times|E^\pm\rangle = E|E^\pm\rangle \qquad 0 \leqslant E < \infty \tag{A.16}$$

and fulfilling outgoing (−) and incoming (+) boundary conditions (e.g. the solutions of the Lippmann–Schwinger equations). For these generalized eigenvectors, we have $|E^{\pm}\rangle \in \Phi_{\mp}^{\times}$ but $|E^{\pm}\rangle$ are not elements of the dual of the Schwartz space $\Phi^{\times}$. There are many other examples of generalized vectors that are in $\Phi_{\pm}^{\times}$ and not in $\Phi^{\times}$. For example, the generalized eigenvectors of the self-adjoint Hamiltonian H with complex eigenvalue, the Gamow vectors

$$H^{\times}|E_R - \mathrm{i}\Gamma/2^{-}\rangle = (E_R - \mathrm{i}\Gamma/2)|E_R - \mathrm{i}\Gamma/2^{-}\rangle \tag{A.17}$$

are elements of $\Phi_{\pm}^{\times}$, but not elements in $\Phi^{\times}$.

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